Preface

The Research Training Group 1493 "Mathematical Structures in Modern Quantum Physics" supports doctoral students and young postdoctoral researchers who work on mathematical and physical questions at the interface between quantum field theory and mathematics.

Modern mathematics is greatly influenced and stimulated by the quest for understanding the physics of space, time and matter at the quantum level. The theoretical treatment of the phenomena observed in that regime requires novel concepts and methods from several branches of mathematics, ranging from new approaches to geometry and topology through innovative techniques in the analysis of infinite systems up to the theory of invariants of operator algebras and category theory. Progress in these areas is important to formulate and apply the ultimate natural laws that govern fundamental physical processes ranging from the genesis of particles in modern accelerators to the formation and development of our universe.

The work on this doctoral students' newspaper was intended to help the students in the Research Training Group to learn what their fellows are working on. The students and postdocs were divided into pairs, which interviewed each other about their research interests and then wrote about each other. Some students who were just beginning preferred to interview one of the faculty members instead.

Presenting mathematical results to the general public is a challenge even for established researchers. This project also had the purpose of training the students for this difficult task. Some students tried out their journalistic talents and wrote articles suitable for laymen without mathematical background. The main target audience, however, were the other graduate students in the Research Training Group. Accordingly, several of the articles assume some familiarity with mathematical and physical notions. Daniel Pape and I edited some of the articles later to make them somewhat more suitable for a general audience.

I particularly thank Daniel Pape for his effort to assemble and edit the following articles. The pictures were taken by Antje Nücklich.

Written by Ralf Meyer.

Picture Gallery


Johannes Aastrup


Vadim Alekseev


Sara Azzali


Dorothea Bahns


Detlev Buchholz


Daniela Cadamuro


Dzmitry Dudko


Alessandro Fermi


Łukasz Grabowski


Alexander Kahle

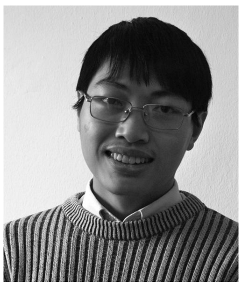

Holger Knuth


Manuel Köhler


Ralf Meyer


Antonia Miteva


Huong Nguyen


Nhu Thang Nguyen


Marc Palm


Daniel Pape


Karl-Henning Rehren


Sutanu Roy

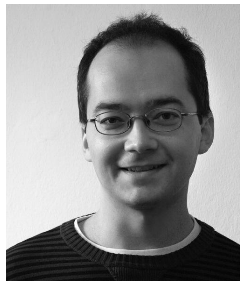

Ko Sanders


Thomas Schick

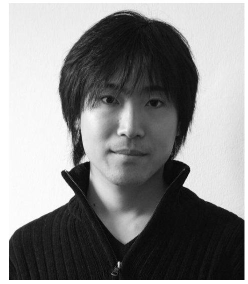

Peter Schlicht


Ingo Schröder

Christoph Solveen



Adrian Szawlowski



Mitsuharu Takeori

Varun Thakre

Nils Waterstraat

Jiguang Zheng

Chenchang Zhu

Seiberg–Witten Invariants for Spectral Triples

The research of Vadim Alekseev

Vadim Alekseev, doctoral student of Ralf Meyer, is working in the area of noncommutative differential geometry, which generalises Riemannian geometry. He studies so called Seiberg–Witten invariants for spectral triples. They have important applications in mathematical physics and geometry.

Noncommutative geometry tries to translate topological or geometric properties into the language of operator algebras. The starting point of this subject is a deep result of Israel Gelfand from the 1940s, which provides a far-reaching generalization of Fourier analysis. To a compact space X, we associate the space of continuous functions $X \to \mathbb{C}$. Endowed with the supremums norm, this space carries the structure of a so-called C*-algebra. This C*-algebra is commutative and has a unit element. The Gelfand–Naimark Theorem states that every commutative C*-algebra with a unit is exactly of this type, that is, there is a one-to-one correspondence between unital commutative C*-algebras and compact spaces. More precisely, for a commutative unital C*-algebra, a point of the underlying compact space is a character or, equivalently, a primitive ideal. Beyond this important example, there is a large dictionary between the topological properties of a space and the algebraic-analytic properties of the related algebra. Mathematicians consider such analogies or classifications as the most beautiful aspect of their field. For noncommutative C*-algebras, we can still define a set of primitive ideals and give it a topology, but the space need not satisfy natural separation axioms. Since the Riesz–Markov Theorem asserts that the linear functionals on a unital commutative C*-algebra can be viewed as some specific type of measure, we also obtain a notion of noncommutative measure theory.

A similar interaction occured between the theory of spectral triples and Riemannian manifolds. For two objects to have the same geometry is in a sense much more restrictive than to have the same topology, hence the information on the algebraic-analytic side has to become more involved. The notion of a spectral triple originally was motivated by mathematical physics. More precisely, they are applied in general relativity and various quantum field theories. A spectral triple $(A, \mathcal{H}, \mathrm{D})$ is a C*-subalgebra A of the bounded operators on the Hilbert space $\mathcal{H}$ together with a selfadjoint, unbounded operator D with discrete spectrum that commutes with all elements of A up to a bounded operator. Alain Connes showed that one can recover from such a triple the smooth metric of the compact Riemannian manifold. Again, we have a one-to-one correspondence between the two theories. Hence, there is a good candidate for a suitable notion for noncommutative Riemannian geometry, namely, spectral triples where the subalgebra A is no longer commutative.

In exactly this abstract setting, Vadim studies a special kind of geometric invariants, named after the physicists Nathan Seiberg and Edward Witten. They were introduced in order to study solutions to Yang–Mills equations and are originally defined only for smooth, compact four-manifolds, namely, as the space of solutions to the Seiberg–Witten differential equations, which is a compact space. There are various reasons why one wants to understand those invariants in a deeper way. One of their main features is that these invariants are able to distinguish smooth structures on Riemannian manifolds.

In his research, Vadim applies various methods coming from K- and KK-theories, gauge theory, operator algebras and index theory to define and study those invariants in the noncommutative case. He tries to obtain results about which information of the spectral triple they preserve. He hopes to obtain an understanding what smoothness in the noncommutative world really means.

Written by Marc Palm.

Azzali developing new ideas in Index Theory

The research of Sara Azzali

Sara Azzali is a post-doctoral researcher at the University of Göttingen who is working in index theory, which applies analytic invariants to study the geometry of manifolds, uncovering deep connections between analysis and geometry.

After a series of articles by Atiyah and Singer on index theory, the heat kernel technique, a new different method for proving the index theorem was discovered. This approach produced many new results and revealed new interesting invariants, which are originally analytic constructs, but turned out to be very useful geometric invariants as well. Sara Azzali has recently been studying such kinds of invariants to distinguish geometric properties of manifolds. These invariants are called eta form and torsion form because they are differential forms on the parameter space when the family of manifolds is described using a parameter space. These invariants contain a lot of important geometric information about manifolds, especially in the case of families of manifolds. They also play fundamental roles in theoretical physics; for example, the adiabatic limit of the eta invariant is related to global gravitational anomalies, a connection first studied by Witten. The invariants are so complicated, however, that we are hardly able to compute them, even in nice cases like well-studied fiber bundles. As a result, it is difficult to find simple examples to examine the expected solutions.

Despite the difficulty, Sara is trying to define and apply these invariants to foliations, which are more complicated geometric objects for studying manifolds, given by integrable subbundles of tangent bundles. In the case of foliations, there are two main difficulties. First, due to the lack of a parameter space, it is difficult to define eta and torsion forms on foliations. Second, since their leaves are not compact, the operators have continuous spectrum, so that it is more complicated to prove the convergence of eta and torsion form. To approach these problems, particularly the second problem, Sara at first dealt with an important example, namely, families of coverings, without assuming any hypotheses on the spectra of the operators, and then could successfully define eta and torsion forms on them. Now she attempts to define eta and torsion forms on foliations using the analogy with this example.

Index theory shows us a beautiful and deep connection between analysis and geometry, which has attracted many mathematicians into this field, among them Sara Azzali. I hope her work will have a large impact on this field.

Written by Mitsuharu Takeori.

Quantum Field Theory on the Moyal Plane

The research of Dorothea Bahns

Scientists dealing with the Quantum world manoeuvre in dimensions that cannot be detected by naked eye. Dorothea Bahns, having been fascinated by this abstract world has tried to fathom the basic structures and laws of the Quantum World . . .

The 33-year-old enthusiast now leads the research group "Non-commutative geometry and mathematical physics." Her research is aimed especially towards the understanding of Quantum Field Theory.

Her main interest lies in understanding the Quantization of Geometries. In her research she mainly focuses on the Geometry of Space-Time, that is, the four-dimensional space, which includes the three spatial dimensions of length, width and height and the time dimension. "A natural and a physically meaningful way to define a quantized version of this, is to use the language of non-commutative geometry," she says. Let us very briefly understand the mathematical meaning and physical motivation behind this.

A theorem by Gelfand states that there is a one-to-one relation between topological (locally compact, Hausdorff) spaces and commutative C*-algebras (that is, algebras with a certain additional structure). A theorem by Connes states that certain commutative algebras with a lot of additional structure (so-called spectral triples) are in one-to-one relation with compact Riemannian manifolds. Hence, geometry is encoded by an algebra. The idea of non-commutative geometry is to consider non-commutative algebras as "non-commutative geometric spaces."

One feature of non-commutative geometry is that the concept of a point loses its meaning. To study geometry without points has a direct motivation in physics, in the sense that it seems to be impossible to define a point operationally (that is, by a measurement).

An event is usually defined to be a point in Space-Time. However, if one wants to give an operational meaning to the notion of an event, then measuring an event in space-time, up to an arbitrarily high precision, would need an arbitrarily high amount of energy, located in an arbitrarily small region in space-time. By the theory of general relativity and quantum theory (assuming of course, that both are valid at small scales), this would form a horizon hiding the very region we are trying to observe! It seems that uncertainty relations hold which restrict the accuracy up to which the coordinates of an event can be measured. One way to implement such uncertainty relations is to assume certain commutation relations between space-time coordinates. Such relations then define a concrete model of a non-commutative space(-time).

Dorothea Bahns is studying how Connes's framework can be generalized to incorporate different concrete models of noncommutative spaces which stem from ideas rooted in physics. The two main difficulties here are that a physically meaningful space-time is neither compact nor Riemannian. Also, she is very interested in defining quantum field theory on concrete models for non-commutative spaces, mostly the non-commutative space defined by the Weyl algebra – already this simple model entails a wealth of interesting consequences.

Why would it be interesting to study quantum field theory on a non-commutative space? Again, it is the fact that a non-commutative geometry is a geometry without points, as we shall see. We start by asking: What is renormalization and why is it needed? The main building block of the perturbative approach to quantum field theory is a fundamental solution (called Feynman propagator in the physics literature) of a hyperbolic partial differential operator that governs the dynamics of that field theory.

The combinatorics of the Feynman graphs yield certain expressions where typically, products and convolutions of the Feynman propagator appear. Now, since fundamental solutions are distributions, the term "product" requires some explanation. But we are quite safe, since in fact, by a theorem of Hörmander, it can be shown that products of the Feynman propagator can be defined (as the pullback of the tensor product along the diagonal map) on any test function that does not have the point zero in its support. This is not trivial as the singular support of the Feynman propagator is much larger than this single point (it is the boundary of the light-cone). Renormalization then is the extension of this product to a distribution that is defined on all test functions.

Here is where non-commutative geometry enters: once one has an idea of how to even define perturbation theory on a non-commutative space (which is far from trivial), there is some hope that, since there are no points, singularities in zero might be absent. While this naive hope has been disappointed in most (but not all!) quantum field theoretic models on the Weyl algebra, it might still be true in general on more complicated non-commutative spaces – and this conjecture is part of Dorothea Bahns' ongoing research.

Moreover, the general renormalization theory of quantum fields on the Weyl algebra remains to be fully understood (in the hyperbolic case), and she spends a lot of her time thinking about this. The relevant distributions – in general, they are more complicated objects than just (products of) Feynman propagators - seem to be elements of the dual space of a certain space of functions of Gelfand–Shilov type.

A somewhat complementary approach to defining non-commutative geometry may be applied to minimal surfaces. Similarly to the general idea of non-commutative geometry, the starting point is the fact that there is a one-to-one relation between minimal surfaces and certain commutative (Poisson-) algebras. A deformation process associates a non-commutative algebra to such a Poisson algebra – and again, this non-commutative algebra defines a non-commutative minimal surface. The biggest open question here is the representation theory of the associative algebra.

Another way to go about this is the Lax pair construction for a certain class of maps. In collaboration with Giovanni Landi, she has almost successfully made this construction for non-commutative sigma-models.

Written by Varun Thakre.

Quantum Field Theories with Factorizing Scattering Matrices

The research of Daniela Cadamuro

Daniela Cadamuro is studying some aspects of the model of quantum field theories with factorizing scattering matrices that were developed recently by Gandalf Lechner.

Quantum field theory is the general framework for the description of the physics of relativistic quantum systems with infinitely many degrees of freedom. There are several approaches, each of which focuses on some particular aspect or property of the theory one wants to construct, and the algebraic approach emphasizes particularly the role of algebraic relations among observables, which determine the physical system itself.

Unfortunately, one of the biggest problems in Algebraic Quantum Field Theory is the explicit construction of a field theory in the presence of interactions. Earlier examples of a field theory with particle interaction, satisfying all the features of an AQFT, namely the axiomatic description given by Wightman, were due to Glimm and Jaffe who model the so called $P(\phi)_2$ models in $1+1$ dimensions and ϕ^4_3 in $2+1$ dimensions.

Recently, Gandalf Lechner has come up with the idea of deforming the fields to construct a nontrivial theory with interaction. More precisely, he deformed the Minkowsky space-time, introducing a deformation parameter for the coordinates. This deformation parameter enters the definition of the fields and their algebra, and the resulting theory is shown to satisfy many of the features of a quantum field theory, that is, the Wightmann axioms, except for the localization axiom. Indeed, due to the deformation itself, these fields are only relatively localized on wedge-shaped regions of the Minkowski spacetime.

Nevertheless the main aspect of such an approach is that the theory so obtained, regarded as a wedge-local quantum field theory on the "commutative" two dimensional spacetime is an example of a model with factorizing S-matrix (scattering matrix), whose two-particle S-matrix is non-trivial. For physical reasons one is interested in considering observables with sharper localization properties than wedge-shaped localization.

The S-matrix is indeed a pure phase, but it does not ensure the existence of observables localized, for example, in intersections of two wedge-shaped regions. Sufficient conditions in AQFT for the existence of such observables are the so-called "modular nuclearity condition," introduced by Buchholz, D'Antoni and Longo, and the "split property for wedges," which ensure the non-triviality of the algebra of observables localized in the intersection of a left wedge with a right wedge. Lechner proved these two properties for some of the models of quantum field theories with factorizing S-matrices.

Daniela Cadamuro is focusing on aspects of quantum field theories with factorizing S-matrices developed by several authors, starting with ideas of Schroer and Wiesbrock, and more recently by Buchholz and Lechner, and she is trying, in particular, to give a more explicit description of the observables in the intersection algebra of wedge-regions for the class of S-matrices for which the modular nuclearity condition and the split property for wedges ensure the non-triviality of this algebra.

The goal of such a construction is very similar to a constructive approach to quantum field theory, namely, the Form Factor Program. This program aims at constructing a quantum field theory in the Wightman axiomatic framework from the input of a factorizing S-matrix. More precisely, fixing an S-matrix, the Form Factor Program would like to construct the n-point functions of a Wightman quantum field theory, which has S-matrix corresponding to the one already chosen. Indeed, would one know the n-point functions of an observable lying in the intersection algebra, then one could recover the action of the local operator itself in the Hilbert space by applying the Reconstruction Theorem of Wightman. The construction is carried out on the space of scattering states, using a certain algebraic structure of the states, namely, the so called Zamolodchikov–Faddeev algebra, and calculating the so called form factors for certain local operators. The technical problem in this approach is to control the very bad divergencies that appear.

Schroer has suggested another procedure to construct a quantum field theory from a factorizing S-matrix. In constrast to the Form Factor Program, the approach of Schroer starts with an appropriate spacetime interpretation of the vacuum representation of the Zamolodchikov–Faddeev algebra, and it follows that the resulting fields can be thought of as wedge-localized. One obtains therefore a local quantum field theory by taking intersections of the wedge algebras generated by the Schroer's fields and the non-triviality of these intersection algebras is again ensured by the modular nuclearity condition and the split property for wedges. Once again the main technical problem in this method is due to infinite sums used for constructing these fields.

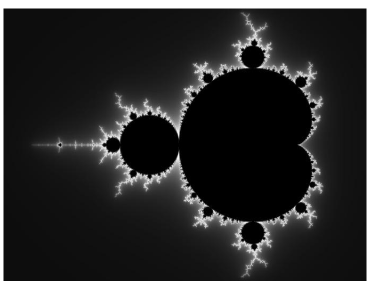
Daniela Cadamuro is trying to combine the two approaches mentioned above to obtain a more explicit description of the observables in the intersection algebras. Moreover, she is interested in understanding the consequences of the nuclear modularity condition on the form factors of observables.

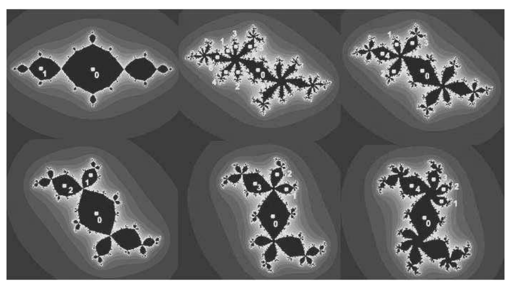
Written by Alessandro Fermi.

The Decoration Conjecture on the Mandelbrot Set

The research of Dzmitry Dudko

Have you ever seen such beautiful pictures? These images are known as the Mandelbrot set and the Julia set, two of the most interesting objects in complex dynamic and fractal geometry. We interviewed Dzmitry Dudko, who does research on complex dynamic. He told us a really enjoyable story about his topic on Mandelbrot set and an open conjecture in this theory.





Dzmitry has learned complex dynamics for three years, and he started his project in October 2008. He is really attracted by the Mandelbrot set, which is the set of all the complex numbers c such that the sequence defined recursively by

$$z_0 = 0, \qquad z_{n+1} = z_n^2 + c \quad \text{for } n \geq 0$$

is bounded. Although this definition seems very simple, the Mandelbrot set has a wonderful structure and many amazing properties. For example, the Mandelbrot set is not strictly self-similar but quasi-self-similar. Moreover, it is a compact connected set, contained in the closed disk of radius 2 around the origin. Mathematicians have been working hard for a long time to prove a major conjecture, which states that the Mandelbrot set is locally connected. It is clear that this question is too difficult to prove directly, but we can try to work on some related questions. His research is to investigate the so-called decoration conjecture: "For every fixed $\varepsilon > 0$, there are at most finitely many connected components of the set difference $M - S$ with diameter greater than ε, where M

is the Mandelbrot set and S is a small copy of it." If the decoration conjecture is not true, then the Mandelbrot set cannot be locally connected. In order to prove this statement, we need to construct a suitable dynamical model and then to use the method of complex dynamics to investigate the Mandelbrot set. Constructing the right model is the main difficulty, but Dzmitry believes that it will be solved soon. We would like to thank him for the interesting interview.

Written by Nhu Thang Nguyen.

Equivariant Index Theory for foliations

The research of Alessandro Fermi

Alessandro Fermi is studying some aspects of equivariant index theory for foliations, using new techniques offered by Alain Connes' Noncommutative Geometry.

The index of an elliptic differential operator on a compact manifold without boundary is an analytic object, defined as the difference of the dimensions of the kernel and cokernel of the operator, which are both finite. Differential operators which have a physical meaning, such as the Laplacian and the Dirac operator, belong to this class of operators.

The main question in index theory is how to compute the index of an operator. A milestone in mathematics (and also in physics) is the celebrated Atiyah–Singer Index Theorem, according to which the index of an elliptic operator equals some characteristic classes of the manifold, showing that the index depends only on the differential topology of the manifold. In other words, this theorem is a bridge between two apparently distant branches of mathematics, namely analysis and topology. Surprisingly, topological information implies analytic properties, such as a direct method to analyze operators, also of physical importance. The converse is also true: analytic features of operators on the manifold yield topological properties of the manifold itself. Consider the case where the $\hat{A}$-genus of a manifold is rational, for example; then the index theorem automatically implies that the manifold is not of spin-kind.

A first generalization of the geometric setup considered above is to consider a fiber bundle together with an elliptic differential operator in each fiber; a family of elliptic differential operators. Under some conditions, the corresponding family of kernels and cokernels turns out to be vector bundles of finite rank over the base manifold and one defines in this case the analytic index of the family is the difference of these bundles in the topological K-theory of the base manifold.

How may we compute the index of such a family? There are several methods, one of which splits our "composed" object into several invariant numerical components and analyzes them.

We would like to generalize this families index theorem further to foliations instead of fibrations. A foliation of a manifold can be described as a distribution on it that satisfies the Frobenius theorem, that is, as a completely integrable sub-vector bundle of the manifold's tangent bundle. This notion is closely related to the notion of dynamical system on the manifold.

But some problems arise from the construction: in general there is no standard fiber for a foliation and, moreover, the "base manifold" of the foliation, called the space of leaves, often has singularities.

The solution of this problem is essentially to enlarge the point of view and to consider the foliated manifold as a noncommutative generalized fibration, to whose basis are associated certain classes of non-commutative algebras. We construct the C*-algebra of the foliation as the non-commutative algebra of the generalized family of pseudo-differential operators of negative order on the smooth holonomy groupoid, made up by unwrapping each leaf of the foliation. Then, for certain classes of suitably defined (elliptic) operators on the holonomy groupoid, one can well-define its analytic index as an element of the K-theory group of the above C*-algebra.

But how may we compute such an index class? One can define a purely topological counterpart of such an index and then show that they are the same as elements in the same K-theory group. This equality is the celebrated Connes–Skandalis Index Theorem for Foliations.

The idea is now to extract topological numerical invariants from the index class, as in the family case, via the cyclic cohomology of the algebra of the foliation. The procedure due to Alain Connes consists in extracting numerical quantities (complex numbers) from K-theory classes via a cyclic cocycle and ends up with the task of computing in topological and geometrical terms the corresponding quantity for the index class.

In particular, Alessandro Fermi is studying the case where the foliated manifold is endowed with the action of a compact Lie group of diffeomorphisms of the manifold preserving the leaves. In this case, one can again define two index classes equivariant with respect to acting group, namely, a topological class and an analytic one in the equivariant K-theory of the foliation algebra and prove an equivariant Connes–Skandalis Index Theorem, that is, the corresponding equality of these two index classes. The equivariant Connes–Skandalis Index Theorem is due to Benameur.

Alessandro Fermi is interested in defining numerical invariants in the case of equivariant K-theory as above, corresponding to interesting geometrical cases, and moreover in computing the numerical quantity of the equivariant index class in topological and geometrical terms, taking also into account the new internal symmetry represented by the Lie group action. He is also concerned with verifying that such a number is an invariant depending only on the fixed point manifold of the Lie group action on the foliated manifold, proving that the nature of numerical quantity of the equivariant index due to the inner symmetry given by the Lie group is in a certain sense a local invariant. The corresponding formulas would be generalizations of higher Lefschetz Fixed Point Theorems in the setting of foliated manifolds.

Written by Daniela Cadamuro.

The Atiyah conjecture for hyperbolic groups

The research of Łukasz Grabowski

Łukasz Grabowski is investigating the Atiyah conjecture for hyperbolic groups for his PhD, on which he started working in October 2008. Already he has contributed to this field of research by introducing concepts from Turing machines in order to simplify the construction of counterexamples (for non-hyperbolic groups) and to obtain sharper results on their properties. Łukasz is somewhat sceptical about the validity of this conjecture for all hyperbolic groups, because almost every discrete group is hyperbolic (in a precise mathematical sense). On the other hand, hyperbolic groups are in some ways very similar to free groups, for which the Atiyah conjecture is known to be true.

Łukasz Grabowski bursts out in laughter when asked how he would explain his research to his grandmother. As a pure mathematician he knows how difficult it is to explain the abstract entities that occupy his daily work to a layman. After some hesitation: "I think I would wrap it up in a single sentence: I work on some far fetched generelisations of algebraic geometrical objects."

In fact, Łukasz works on the Atiyah conjecture for hyperbolic groups, named after Sir Michael Atiyah, a 1966 Fields medallist and winner of the 2004 Abel prize. In 1976 Atiyah considered square-integrable differential forms on a Riemannian manifold with a group action and wondered if the dimensions of their cohomology groups, the so-called L^2-Betti numbers, which are a priori real numbers, can be irrational. The rationality of these numbers became known as the Atiyah conjecture. Nowadays, the conjecture is often formulated in an equivalent, purely group theoretical setting.

Łukasz explains it again from the very beginning. Consider the group of integers and the associated group ring, which we can interpret after a Fourier transformation as the field of rational functions on the circle. In algebraic geometry a geometric object is described by the (algebraic) functions on it, so in this way the circle is related to the group of integers. The Atiyah conjecture for this case says exactly that the group ring can be embedded into the field of Laurent polynomials. A similar argument works for all Abelian, finitely generated groups without torsion. For non-Abelian groups the Atiyah conjecture becomes more abstract, because there is no Fourier transformation or dual group. In a sense, it is the starting point of non-commutative algebraic geometry.

Łukasz came to Göttingen in October 2008 to do his PhD, after finishing his studies in Szczecin, Poland. He is working with Andreas Thom, who recently moved to Leipzig. For the moment Łukasz is staying here and contacts his supervisor via many emails. He explains the importance of the hyperbolic groups which are the subject of his research: “If you randomly choose a discrete group, it is hyperbolic with probability one. In this sense, almost all groups are hyperbolic, although many explicit examples are special and not hyperbolic. Proving the Atiyah conjecture for hyperbolic groups would be a rather strong result. This is also the reason why I am sceptical about whether it is true at all, even though hyperbolic groups have some properties that make them somewhat similar to free groups, for which the Atiyah conjecture is known to hold.”

Recently Łukasz’ interest has shifted to counterexamples to the Atiyah conjecture. For torsion-free groups or hyperbolic groups there are no known counterexamples, he explains, but for non-hyperbolic groups with torsion there are. This was proved in 2009 by Tim Austin, based on earlier work of several other people including our own Thomas Schick. Interestingly, the mathematical techniques for finding counterexamples are very different from the ones used to prove the Atiyah conjecture. Łukasz: ”People use, for example, ideas from the theory of dynamical systems. My own contribution has been to simplify some lengthy computations in the known constructions by introducing a different conceptual framework based on Turing machines. This also enabled me to prove that all real numbers can appear as L^2-Betti numbers and all computable numbers already appear for finitely presented groups.”

This is in fact what Łukasz likes best about scientific research: having your own idea and being able to work on it. Having an idea is difficult, he admits, but it is very rewarding to see things work in the end. Although he believes that practical applications of his research are far away, he does hope that his ideas may be of interest for people working in computability theory. We wish him good luck during the remainder of his time in Göttingen and many fruitful ideas.

Written by Ko Sanders.

The universal coefficients theorem for equivariant KK-theory

The research of Manuel Köhler

Manuel Köhler works in the field of non-commutative geometry on so-called K- and KK-theories, which are used to describe invariants on non-commutative spaces. An important result about KK-theory is the Universal Coefficient Theorem for non-equivariant KK-theory. Manuel wants to show corresponding statements for equivariant theories, in which a group action encodes the symmetry of a non-commutative space within the K- and KK-theories, for certain groups. This makes it possible to calculate the KK-theory, when the simpler K-theory is known.

A manifold – studied by classical geometry – can also be described by a commutative algebra of functions on this manifold. Concepts of geometry, as for example the distance between two points, can be transferred onto this algebra of functions. Once these geometrical concepts are formulated for commutative algebras, one can also define them for other algebras, which are non-commutative. This way, so-called non-commutative geometry is described and one gets to non-commutative spaces.

Other concepts can be transfered from manifolds, that is, commutative spaces. When some category, for instance, all manifolds, is divided into interesting classes, for example, using topological properties, it becomes useful to look for entities that are invariant within these classes. A suitable invariant in topology is the fundamental group. K-theory is another invariant that associates abelian groups to non-commutative spaces looking at vector bundles on these spaces. This theory is relatively well understood.

KK-theory is another example of an association of abelian groups to non-commutative spaces. It is a very flexible and general generalisation of K-theory that uses differential operators. This makes the theory conceptionally very appealing, but also very complicated, as concrete calculations are concerned. Thus it is very interesting to reduce calculations in KK-theory to calculations in K-theory. This job is done by the Universal Coefficients Theorem (UCT). It states a short exact sequence, which is a chain of maps of the form

$$0 \to A \to B \to C \to 0,$$

where the image of one map is always the kernel of the next one. Here the KK-group stands in the middle and groups of morphisms between K-groups are the objects on both sides. The KK-theory is expressed in terms of K-theories by this exact sequence. This was shown for so-called non-equivariant K- and KK-theories.

Manuel works on the UCT for equivariant K- and KK-theory, where an additional symmetry group is taken into account. It is known that the UCT does not hold for all groups. While the case of the group $\mathbb{Z}$ was already shown before, he could show the UCT in a slightly different formulation up to now for the groups $\mathbb{Z}/p\mathbb{Z}$ with p a prime number ≤ 19. He expects it to be true for all p, but for the higher p some technical difficulties still have to be resolved. Looking out further he aims at a slightly weakened form of the UCT for all finite groups.

The step from non-equivariant to equivariant theories is not at all straightforward. Manuel uses instead of the original analytical approach of KK-theory a new algebraic, categorial one, which makes the proof of the UCT in equivariant cases feasible by translating the problem into one in a completely algebraic setup. This very abstract approach to KK-theory contains some algebraic objects, which are not understood up to now and raise the biggest difficulties. These objects are connected to problems of number theory like Fermat's last theorem. So it combines very different subjects within mathematics, which need to be understood.

Such a Universal Coefficient Theorem reduces calculations in KK-theory to the much simpler K-theory. Moreover, in cases with the same K-theory the KK-theory is also the same. For this reason, the UCT is an essential ingredient in the classification of a certain class of C*-algebras.

Written by Holger Knuth.

Conformal Quantum Field Theory, Correlation Functions and Supersymmetry

The research of Holger Knuth

Quantum field theories provide a theoretical framework for constructing quantum mechanical models of systems which are classically described by fields. They are equipped with a symmetry, encoded by a group action. One output of a quantum field theory are correlation functions, they are of theoretical as well as practical importance.
Holger Knuth works in the context of conformal, supersymmetric quantum field theories. He wants to express a specific class of correlation functions in terms of a few elementary functions.

Holger Knuth is a tall, long-haired physicist who just started the third year of his Phd studies. To our meeting, he comes equipped with three papers related to his project and as the interview starts, he unfurls one of them on the table and we plunge straight into the realm of quantum field theory:

Quantum field theories yield a mathematical language to describe and analyse the physics of elementary particles. The standard approach to quantum field theory is via the action principle which originates in classical physics. The classical action principle gives a way to derive differential equations, the so called field equations, to describe how fields behave in a given physical context. There is a process to "quantize" these fields to get a quantum field theory from of a classical field theory. However, this approach yields results which agree with experimental data only in situations where there is no strong interaction between the particles which are to be described. For example, description of quarks, the elementary building blocks of neutrons and protons, via the standard approach is restricted to situations with a high energy level. It describes physics "in large particle accelerators such as LHC at CERN not the physics in this table" as Holger Knuth puts it and knocks on the table on which I am taking my notes.

Another approach to quantum field theory, which is more appropriate for situations with strong interaction between particles focuses on the so-called correlation functions. In this approach, a field is described by an operator valued distribution, the operators act on a Hilbert space H with a distinguished vector Ω, the so-called vacuum state. More sloppily, one may think of functions ϕ that assign to each point x of a given underlying space X an operator $\phi(x)$. An n-point correlation function is an assignment of the form

$$(x_1 \ldots, x_n) \mapsto \langle \Omega | \phi_1(x_1) \ldots \phi_n(x_n) | \Omega \rangle.$$

Correlation functions are the output of the theory that is actually physically measurable, for example the probabilities in a scattering experiment may be described by four-point correlation functions. Correlation functions are also of theoretical importance, being the fundamental building blocks in the approach to quantum field theory via the so called Wightman axioms.

In a variant of quantum field theory, which is close to the one Holger Knuth is working with, the underlying space X is four-dimensional Minkowski space-time, that is, $\mathbb{R}^4$ equipped with the Minkowski metric. Symmetries of the underlying space are a fundamental concept to understand quantum field theories. The correlation functions are restricted by the symmetry of the underlying space, formalized by a group action on it. A conformal quantum field theory, for example, is equipped with an action that preserves angles defined via the Minkowski metric. Supersymmetric, conformal quantum field theory, the theory Holger Knuth is focussing on, has another symmetry given by adding nilpotent variables to the algebra of functions on four-dimensional Minkowski space-time, making it slightly noncommutative. Supersymmetric theories have the advantage that they are more flexible, there is more freedom in describing physical situations since there are more fields due to the larger symmetry group of the underlying larger space.

In the case of four-dimensional conformal field theory without supersymmetry, one is able to write down an expression of four-point correlation functions, which are invariant under the conformal group action, in terms of two independent elementary functions of four points also being invariant under this group action. In an expansion of the scalar four-point correlation functions (that is, correlation functions of fields without spin) corresponding to the representations of the conformal group, the so-called partial wave expansion, one is able to find closed forms for the coefficients depending only on a few constants. This makes four-point correlation functions much more accessible, in particular it allows for solving a problem related to the verification of one of the Wightman axioms.

Holger Knuth wants to express scalar four-point correlation functions in the supersymmetric case in terms of more elementary functions (or elementary invariants). There are 14 invariants in the nilpotent variables, which in addition to two invariants known from the non-supersymmetric case, are enough to express all (not only the scalar) four-point correlation functions. Holger Knuth was able to show that if one restricts to scalar four-point correlation functions, 8 of the 14 additional invariants suffice. Expanding scalar correlation functions as polynomials of the nilpotent invariants, Holger Knuth showed that there at most 36 nonvanishing products of the 8 nilpotent invariants. He also showed that if one neglects products of higher order than 1, then the expansion can be expressed only in terms to the 2 invariants coming from the non-supersymmetric case. He hopes to show that this reduction also works for higher order products.

Causes of difficulty are that the relevant expressions are quite complex because many nilpotent variables are involved and that calculations are complicated because these nilpotent variables do not commute. However, the complete reduction would be an important step, since it would allow for applying methods from the non-supersymmetric to the supersymmetric case.

After almost three hours of intense discussions, Holger Knuth gets up, kindly offers me his papers as a complementary source to my quickly scribbled notes and heads back to the train station to catch his train to Hannover. I wish him all the best for the remaining year and his work on supersymmetric conformal quantum field theory.

Written by Manuel Köhler.

The Axiomatic Approach to the Micro-World

The research of Antonia Miteva

This is a short interview with Antonia Miteva on her research on theoretical foundations of the physics of tiny things moving at high speed.

DP: What field are you working in?

AM: I am working in axiomatic quantum field theory.

DP: How would you try to describe your work to the layman?

AM: I would say that we want to understand how the Universe is functioning on its lowest level at very high energies – on the level of its finest compounds when they move very quickly. More concretely, it is concerned with combining quantum mechanics with general relativity to achieve such a description.

DP: Do you think it is hard to describe your work to a layman?

AM: It is not possible to communicate our results in their genuine form because they are written in a language that the layman does not understand. So, one needs to make some translation, which is a hard task.

DP: In a previous conversation you explained to me that there are two axiomatic approaches to quantum field theory – a field-theoretical one by Wightman and an algebraic one by Haag and Kastler, which both have advantages and disadvantages ...

AM: Yes, in relativistic quantum theory the central objects are quantum fields, and the physical quantities must be expressed in terms of quantum fields. The Wightman axioms, which are connected directly to the fields, arise naturally. It is natural to formulate general dynamical principles in terms of fields and natural to perform concrete calculations. At the same time, it is technically difficult to work with fields because they are complicated mathematical objects. It was noticed by Haag and Kastler that the relevant physical information is encoded merely in the algebra of fields. Their algebraic-axiomatic approach is beautiful, conceptually clear, intuitive and provides really powerful tools. However, there is the disadvantage that explicit model computations are often too difficult in this framework.

DP: What is the goal of your research?

AM: It is a challenging problem to understand how these two approaches are working together and it is not at all trivial to translate information between them. Therefore, the main purpose of my project is to demonstrate how one combines the two approaches to investigate a concrete model. In general, it is a very hard problem, so first we try to solve it for easier models.

DP: What are the specific simplifications of the model you are working with?

AM: My model is in lower space–time dimensions (two instead of four) and with higher symmetry, the so-called conformal symmetry which extends the relativistic Poincaré symmetry.

DP: Why is your research field of interest?

AM: To understand the interplay of quantum theory and general relativity turned out to be an incredibly hard task which has remained unsolved for more than three quarters of a century and at the same time it is of great physical importance. For example, to understand the micro-structure of the universe we have to perform experiments involving particles at very high energies.

DP: What did you find out?

AM: With the combination of the two axiomatic approaches we were able to investigate the only model from the sample of the so-called minimal models and their extensions which could not be studied with group-theoretical methods. With the algebraic approach we were able to find the possible spaces of physical states, or superselection sectors as we call them, and in the Wightman approach we understood a lot about the structure of the commutation relations among the fields. The commutation relations are important because they carry dynamical information.

DP: What do you want to find out in the nearest future?

AM: The last thing by now which we understood about the commutation relations was the Jacobi identity which arises from them.
As a next step we want to explore the stability of its structure under perturbations, applying some cohomological methods.

DP: Which kind of cohomology do you have in mind?

AM: It seems that it will be Hochschild cohomology.

DP: What are your future plans?

AM: To be physical: There is a huge amount of uncertainty.

Written by Daniel Pape.

Trace Formulas in Analytic Number Theory

The research of Marc Palm

Marc Palm is working in the area of analytic number theory and its connections with non-commutative geometry.

In 2009 mathematicians celebrated the 150th anniversary of one of the main objects in number theory – the famous *Riemann ζ-function*, which was defined by Bernhard Riemann in 1859, while he was a professor in Göttingen. It is defined by the formula

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}$$

for $\operatorname{Re} s > 1$ and can be meromorphically continued to the whole complex plane with a single pole at $s = 1$. It has zeroes at the points -2, -4, and so on, called *trivial*. The *non-trivial* zeroes of the Riemann zeta function govern the asymptotic behaviour of prime numbers. The celebrated *Riemann hypothesis* asserts that all these zeroes have real part equal to $\frac{1}{2}$. It is considered by many mathematicians to be the most important unresolved problem in pure mathematics.

The history of the Riemann hypothesis continues around 1912 and again in Göttingen. Around 1912 George Pólya and David Hilbert suggested that there should be a *spectral interpretation* of non-trivial zeros of Riemann zeta function – the imaginary parts t of the zeroes $\frac{1}{2} + it$ should be eigenvalues of an unbounded selfadjoint operator (of course, that would imply the Riemann hypothesis). At that time there was no indication of where this operator should come from and why this should be true, so it was rather speculative.

However, the situation changed with the discovery of the *Selberg trace formula* in 1956. If we take a compact hyperbolic Riemann surface, then it is isomorphic to the quotient of the hyperbolic plane by some group of biholomorphic transformations. The trace formula relates the spectrum of the Laplacian on the surface to the lengths of closed geodesics, which can be described in terms of conjugacy classes of the transformation group. Being at first sight a geometric result, it strongly resembles *Weil's explicit formula*, which relates the zeroes of the zeta function and prime numbers in a similar way. So one can at least hope to obtain a formula of the form

$$\sum_{\rho} F(\rho) = \mathrm{tr}(F(T)),$$

where T is some naturally arising operator.

Marc's research interests concern several generalizations of these results. One can generalize the Riemann hypothesis by introducing so-called L-functions, which are formally similar to the Riemann zeta function, and asking the same question about their zeros, including a spectral interpretation. These L-functions also play a key role in the Langlands program, which is a series of conjectures connecting them to representations of certain groups (the so-called automorphic cuspidal representations) and therefore also giving an analytic interpretation of number theoretic data. The Arthur–Selberg Trace Formulas, which generalize the Selberg trace formula, often help to prove the Langlands conjectures in some cases.

Written by Vadim Alekseev.

Obstructions to non-negative scalar curvature

The research of Daniel Pape

Daniel Pape is working in the group of Thomas Schick since April 2008 on higher index theory. His task is to reprove some results of Gromov and Lawson about obstructions to positive scalar curvature in a "nicer" setting, using Roe's coarse index theory and ideas of Fomenko, Mishckenko and Rosenberg.

An interesting problem in mathematics is which smooth manifolds may support a metric whose scalar curvature is everywhere non-negative. A closely related problem is which functions may appear as the scalar curvature of some Riemannian metric. Once the function has a single negative value and the dimension is at least 3, then such a metric exists. However, the case of non-negative scalar curvature remains a challenging open problem.

Daniel's task is to explore situations where non-negative scalar curvature is impossible, that is, to find obstructions to non-negative scalar curvature for a certain class of manifolds.

The first step in this field was a seminal idea of the french mathematician André Lichnerowicz in the 1960s. He adapted the Dirac operators from quantum mechanics to even-dimensional compact spinable manifolds. By a witty trick he showed that the scalar curvature sign problem is connected to the index of the positive part of the Dirac operator. Namely, if this index is non-zero, then there is no metric with non-negative scalar curvature. The index of an operator A is the difference of the dimensions of $\ker A$ and $\ker A^*$. The topological equivalent of the Lichnerowicz index obstruction is the topological $\hat{A}$-genus obstruction. Here $\hat{A}$ is a certain characteristic class of the tangent bundle of the manifold.

In Lichnerowicz' obstruction indices were positive integers. In the 1970s, Atiyah introduced for the first time a so-called higher index, that is, an index with values in the K-theory of some C*-algebra.

In the 1980s, this idea was developed further by Gromov and Lawson, who introduced the notion of enlargeability of a manifold, which allowed then to apply the Lichnerowicz trick to the Dirac operator coupled to the fundamental group of the manifold. They proved that, under certain assumptions, obstructions for a manifold M can be found on submanifolds N with codimension 2, that is, such that $\dim M - \dim N = 2$. It is remarkable that properties of such a submanifold restrict so much the features of the big manifold. A drawback of the setting of Gromov and Lawson is that it involves incomplete and non-compact manifolds.

The goal of Daniel's research is to reprove parts of the result of Gromov and Lawson in a nicer setting. Again, main ingredients will be the generalized Dirac operator coupled to the fundamental group of the manifold, together with K-theory – but this time K-theory of a Roe C*-algebra, following Roe's coarse index theory and the ideas of Fomenko–Mishckenko–Rosenberg.

It was also necessary to make sure that all the techniques may be combined, especially that coarse indices make sense for coupled Dirac operators and that Roe's index theorem holds. The next issue is to relax the assumptions of Gromov and Lawson on the manifold M and its submanifold N.

Written by Antonia Miteva.

Von Neumann Algebra Methods in Quantum Field Theory

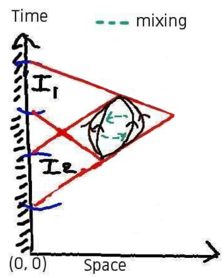
The research of Karl-Henning Rehren

The recent work of Rehren concerns the action of the modular group for algebras in quantum field theory associated with the union of disjoint regions, which has interesting consequences for quantum field theory in a two-dimensional half-space.

The basic structures in quantum mechanics are the algebras of observables (bounded operators in a Hilbert space) and the states (elements of the Hilbert space). In quantum field theory, a field is an operator that depends on space-time. Observables are defined in some region of the space-time. For every region in the space-time the von Neumann algebra generated by the observables are localized in that region. Thus for a given space-time, we have a system of algebras which are the sub-algebras of the algebra of bounded operators on a some separable Hilbert space.

The relations between these algebras reflect the geometric position of the regions. For example, the algebras must commute when there is no communication in terms of signal between two regions, which is an important physical principle known as local commutativity. In addition to that, there are physical principles known as *axioms*, like symmetry, stability (defined as the spectrum of an operator called *energy* which is bounded below). A *vacuum* state is the eigenvector of some specific operator with zero eigenvalue.

The mathematical evolution started with the Tomita–Takesaki Theorem in 1967, which associates to a von Neumann algebra and a state with two special properties – cyclic and separating – an automorphism group of that von Neumann algebra, known as internal dynamics or modular group. *How does the modular group act for algebras of local observables?* In particular situations, where the state is the vacuum and the algebra is associated with a wedge shaped region in a space time of dimension at least 2 which is bounded by planes of light rays, the modular group has a special form, known as geometric action, on the algebra of the observables defined in the regions. In certain types of quantum field theories called conformal, the modular group acts geometrically also for regions, like double cones, for the vacuum state. The orbit of a state, under the action of the modular group is known as *Lorentz boosts*. There is another related effect, known as *Unruh* effect, which says that, if you move in an accelerated way in the empty space the space does not look empty anymore.



The recent work of Rehren concerns quantum field theory in a two-dimensional half-space. Thus we have a boundary which could model or represent a black hole, insulator or metal wall. The algebra for a diamond contains the (or sometimes coincides with) algebras of pairs of open disjoint intervals on the boundary, which may be regarded as the quantum field theory of the one-dimensional space-time. The interesting problem is to find out the action of modular group for the union of these open disjoint intervals.

This problem has been studied independently by Rehren with collaborators Roberto Longo (Rome) and Pierre Martinetti (Göttingen) and by the two mathematical physicists Casini and Huerta from Argentina. Longo and Yasuyuki Kawahigashi (Japan) had found that the modular group is geometric within these pairs of disjoint open intervals, in certain manufactured excited states. Rehren, Longo and Martinetti found that the orbits are different as there is an influence of the boundary. Casini and Huerta computed the modular group of two intervals in a vacuum state considering a special algebra. The action of the modular group for the union of two intervals of interest consists of two things, namely, the geometric action on each intervals and mixing between two intervals. When the modular group acts on an observable in one interval, then the orbit may lie in the other interval with a certain probability, known as mixing. Amazingly, the geometric part is exactly the same in the both works. The result for the union of two open disjoint intervals in the zero-space time dimension (one-dimensional space-time) can be generalized to a union of a finite number of open disjoint intervals, but the geometric picture about diamond regions in the half-space no longer works.

Currently, Rehren and his collaborators are trying to understand the universal behaviour which holds for every general quantum field theory on the half-space. They can show that the geometric modular action cannot be completely universal, because there arise obstructions when the local algebras are "too big" or "too small," but the precise conditions are not yet understood. According to Rehren it is an exciting question whether there is an application to real black holes or Hawking radiation, which is not yet known, and may open future research directions.

Written by Sutanu Roy.

Living on the borderline

The research of Ko Sanders

Do you know this frustrating feeling when you're talking with somebody from Bavaria, and you know that you should understand him, because after all he speaks the same language, but at best you just get a vague feeling for what he's trying to tell you? Or when you go to a talk by a physicist, and all the words he's using are ones of which you know definitions, but *somehow* they seem to be in not quite a good order? Afterwards, it feels so good to go back to standard German, or read a clear-cut definition. Because of that we all know that no-one sane would choose to live in Bavaria, or work in the physics department . . . right?

Meet Ko Sanders, a brave post-doc of 31, mathematician, who chose to do just that: work and talk with physicists, try to understand what they're saying, put it into a meaningful mathematical framework, and – most importantly – prove honest theorems in this framework.

If I had to describe Ko's work in one short paragraph, I would say that he is applying the philosophy of WWGD to physics: if you don't understand something, if it feels as if definitions are wrong, if it seems that something interesting is underneath but you can't quite see it; then ask yourself: What Would Grothendieck Do?

Of course, in physics it is even one step more difficult than in mathematics, because, for example, usually there are no definitions to start with.

Currently, Ko is working on perturbative quantum field theory in curved space-time. In informal terms it can be described as follows: we start with a physical theory which concerns itself with fields of some kind. Sometimes it is relatively easy to tackle, that is, the differential equation whose solutions are fields of interest for the given problem is linear. The term the physicists use in such a situation is *free* field or *non-interacting* field theory.

However, many real-life fields interact with themselves – in mathematical terms it means that the equation is non-linear. Such equations are much harder to solve. The method physicists came up with is the following: write solution to the equation as a "Taylor series" in a variable which is called *coupling constant*, which measures how strongly the field interacts with itself.

In principle, this is a rather clever idea but the problem is that the fields which are of greatest interest are often very pathological (they are not functions but distributions), and as such cannot be multiplied, so that Taylor series have very little meaning.

However, somehow physicists manage to obtain in this way the right answers – as Ko puts it "all the infinities are removed by hand."

Understanding what is really going on can potentially have a great value

– it can both increase our knowledge about mathematical issues related to singular solutions of differential equations, and, well, help understanding the inner workings of the universe. One of the ideas, suggested by Ralf Meyer, Ko is examining right now in this context is the following: instead of using topological completions of the space of well-behaved fields (that is, such which for example can be multiplied), about which we know that it produces ill-behaved fields, use a bornological one. Bornologies are certain structures on sets which allow to talk about boundedness (but not necessarily continuity) of functions – Ko is very optimistic about using this concept.

One should mention that there are already success stories of this "let's axiomatize physics" approach. One of the examples is so called algebraic quantum field theory. Once the axioms of it had been firmly established it became possible to prove theorems which are interesting both mathematically and, after translating it back to physics, philosophically. Ko himself was able to treat in his Ph.D. thesis a "curved space-time" case of the following theorem: no matter how small an open set of the space-time we take, it is possible to obtain almost any state of the universe from the so-called vacuum state just by performing measurements in this small open set.

What is remarkable, although the statement above seems very imprecise, it has both a definite mathematical meaning, and provides an important viewpoint in physics.

Ko hopes that the same kind of utility can be achieved after systematizing perturbation theory. Therefore we can do nothing but wish that Ko – and his co-workers – succeed in their task.

Written by Łukasz Grabowski.

Riemannian Metrics of Positive Scalar Curvature

The research of Thomas Schick

One research direction of Thomas Schick is the existence of Riemannian metrics on smooth compact manifolds with everywhere positive scalar curvature.

The scalar curvature is a real-valued function over a manifold with specified (Riemannian) metric, which can tell us the way this object bends or stays flat locally. For example, taking the following objects as two-dimensional compact manifolds embedded in three-dimensional space, we know that the curvature of the sphere is positive everywhere, while the curvature of the torus depends on the point on the torus: it is equal to zero at points lying on the horizontal red circle, positive at those lying in the horizontal green one, and negative at saddle points.

But if the torus is furnished with the so-called standard metric, it is flat, that is, its scalar curvature is equal to zero. A natural question is, given a connected smooth compact manifold is there a metric g such that the curvature function is positive everywhere?

The answer is No in this two-dimensional case is because of the Gauss–Bonnet Theorem, which states that the Euler characteristic of a manifold (a certain topological invariant) is equal to the average curvature on the whole object. Hence a manifold with negative Euler characteristic cannot have an everywhere positive curvature.

Naturally, we would like to answer the above question for higher-dimensional manifolds by using the same strategy. However, such a No criterion is valid only for the manifolds with a spin structure. In such manifolds there is a so called Dirac operator depending on the metric, which acts on sections of the spin bundle. The Schrödinger–Lichnerowicz formula states that if the curvature is positive, then the Dirac operator vanishes only the zero section. Hence the index of the Dirac operator must vanish. Together with the Atiyah–Singer Index Theorem, which identifies this index with a certain topological invariant, the $\hat{A}$-genus, we conclude that a spin manifold M with non-zero $\hat{A}$-genus carries no Riemannian metric with positive scalar curvature.

Unfortunately, this strategy does not work even on the torus case, whose $\hat{A}$-genus vanishes, and also for some spin four-dimensional manifolds.

Therefore, Schick is studying refined No criteria for the class of spin manifolds. The idea is to make the Dirac operator more complicated, by taking coefficients in a certain C*-algebra. The index of the Dirac operator is generalized to a class in the K-theory group of this C*-algebra. As expected, when this K-theory class is non-zero, then an everywhere positive scalar curvature cannot exist.

The difficult point is to construct an appropriate meaningful characteristic of the manifold. His current research is to identify situations where this index obstruction can really be computed and shown to be non-zero.

Written by Huong Nguyen.

Are Unitarizable Groups Amenable?

The research of Peter Schlicht

It is known since the 1950s that amenable groups are unitarizable. In recent years, there has been progress on the converse question, leading to a geometric perspective on the problem. Peter Schlicht is trying to answer it by studying certain infinite-dimensional spaces with convex metric.

Peter is a member of our Research Training Group for some six months now, and he is working in geometrical group theory, supervised by Andreas Thom. The theory of groups and the study of their representations is vital for many branches of mathematics as well as modern physics. It is therefore desirable to find and relate interesting properties groups may possess with regard to their representations. Peter investigates the relation between two (at first sight unrelated) properties: unitarizability and amenability.

Unitarizability is interesting because unitary representations are well understood and have many nice properties. Therefore, it is very useful if a representation ρ of a group G on some Hilbert space is unitarizable, that is, if there is an invertible operator S on the Hilbert space (a "unitarizer") such that the map which sends a group element g to the operator $S^{-1}\rho(g)S$ is a unitary representation. The whole group is called unitarizable if any of its uniformly bounded representations is unitarizable, where such a representation is called uniformly bounded if it is bounded in the supremum-norm. If this property holds, the study of (uniformly bounded) representations of the group reduces to the study of its unitary representations.

Amenability is another useful property of groups, for instance, in finding group invariant objects by means of averaging. Roughly speaking, a group is called amenable if it carries a kind of averaging operation on the bounded measurable functions that is invariant under translation by group elements. Finite and compact groups are amenable, for example.

In the 1950s, Day and Dixmier proved that amenable groups are unitarizable. They asked whether any group is unitarizable and, if not, whether the converse holds, that is: is every unitarizable group amenable? Some years later examples of non-unitarizable groups became known (like the special linear group of two-dimensional real matrices), so the answer to the first question was negative. Peter tries to answer the second question, which remains an open problem up to the present day.

Peter does not stand alone: important progress has been made in recent years which helped to open new ways to attack the problem. In particular, Gilles Pisier was able to show that for a given group, every of its uniformly bounded representation ρ possesses a unitarizer S with $\|S\| \cdot \|S^{-1}\| \leq \|\rho\|^2$ if and only if the group is amenable. Pisier also showed that for every (uniformly bounded) representation ρ of a unitarizable group there are constants C and α such that for any unitarizer S one has $\|S\| \cdot \|S^{-1}\| \leq C\|\rho\|$.

The difficult part is now to obtain information about these constants. This is desirable because if one is able to show that one may choose $C = 1$ and $\alpha = 2$, Pisier's first result would answer Dixmier's second question.

Hoping to gain new insights, Peter wants to interpret the problem in a more geometric way. He strives to understand unitarizability using the adjoint action of the group on the cone of positive self-adjoint operators on the Hilbert space. This cone is lying in the infinite-dimensional vector space of self-adjoint operators of the given Hilbert space and may be viewed in analogy to a manifold. There is a metric on this cone, which turns out to be convex. Peter was able to re-derive the second inequality above using this picture, in which the supremum norm of a representation is related to the diameter of the orbit of the identity operator. This must be compared with the distance between this orbit and the (convex) set of fixed points in the above cone.

Peter now hopes to obtain information about the possible values of the above constants by deepening his understanding of the geometry of the cone of positive operators. The infinite dimension makes it more difficult to discuss geometry, for instance, it becomes more difficult to define angles. This is interesting in its own right and in studying these questions Peter hopes to contribute to our understanding of spaces with convex metric. Moreover, the connection between the algebraic and geometric point of view seems worth to investigate further in its own right.

Written by Christoph Solveen.

Local Thermal Equilibria in Quantum Field Theories in Curved Spacetimes

The research of Christoph Solveen

Christoph Solveen is looking for local thermal equilibrium states in quantum field theory in curved space-times, following ideas of Buchholz, Ojima and Roos. Just recently, he succeeded in finding states that are pointwise in local thermal equilibrium for a broad class of space-times.

Quantum field theory was established to attempt the unification of quantum mechanics and the theory of special relativity. While in more commonly used approaches to this, physicists use quantum fields as operators on Hilbert spaces obtained by a formal procedure called quantisation, the more abstract Ansatz by Haag and Kastler uses nets of so-called local algebras of observables, which are algebras $A(O)$ associated to space-time regions O in the standard Minkowski space. Their inductive limit is called the quasi-local algebra of observables. Informally speaking, one thinks of a self-adjoint element of such an algebra $A(O)$ as a measuring device localized in the space-time region O. The result of evaluating a state on such an element is considered as the expectation value of a large number of measurements of the observable in the given physical state. The non-commutativity of the local algebras takes into account the fact that quantum effects yield physical quantities that are incommensurable (the Heisenberg uncertainty principle).

Two regions in space-time are called space-like separated if one would have to travel quicker than light in order to get from one region to the other. The principle of locality demands that algebras referring to space-like separated regions commute with one another, so that observables in these regions are commensurable.

The action of the Poincaré group of time-orientation preserving isometries of space-time describes the dynamics and the evolution of time. This allows to define what thermal equilibrium states should be. KMS-sates are special thermal equilibrium states with other characteristics such as the impossibility to generate energy in cyclic processes. Those states are particularly interesting to physicists doing thermodynamics, as those are the situations in which this theory is fully applicable and can describe an entire system by just a few variables such as energy, entropy, temperature, pressure, and so on.

It seems, therefore, sensible to look for states that are at least locally in thermal equilibrium, in order to try to apply thermodynamics locally.

To define such a state, Buchholz, Ojima and Roos considered spaces $Q(x)$ of pointwise localized observables for any point x in Minkowski space. These spaces can be obtained from the net of local observables in a well understood way and turn out to consist of quadratic forms (the pointlike fields). A state is then called $S(x)$-thermal if it looks like a reference state with regard to measurements performed with members of a subspace $S(x)$ of $Q(x)$. The subspace $S(x)$ (the so-called space of thermal observables) has to be chosen such that its elements are sensible to thermal properties of the reference states. The latter are mixtures of global equlibrium states, for which thermodynamics is applicable.

This definition is extended to whole regions of spacetime in the following way. A state is called $S(0)$-thermal, if the above definition is true for all x in the spacetime region O. Having this, it is possible to pointwise associate thermal properties to states sufficiently close to equilibrium. This in turn allows one to determine spacetime evolution patterns of quantities that are relevant in thermodynamics from the dynamics of quantum fields themselves.

One would like to define a notion of local equilibrium in the context of quantum field theories in curved spacetime. This latter framework tries to incoporate classical gravity into the picture by replacing Minkowski space with some (well-behaved, that is, globally hyperbolic) curved Lorentzian manifold. The problem in defining thermal equilibrium in curved spacetimes is the absence of suitable spacetime symmetries. What's more, not even a vacuum state can be defined immediately, as the requirement for this state to look the same for any inertial observer cannot simply be taken over (inertial observers cannot be defined globally any more).

Nontheless, Buchholz and Schlemmer could define local thermal equilibrium on so-called locally covariant quantum field theories. For this one uses “master fields” ϕ that assign to each spacetime (M, g) a quantum field $\phi(M, g)$ which propagates on this spacetime (here M denotes the manifold and g the Lorentzian metric that describes, among other things, the causal structure of the spacetime). These master fields have to comply with certain principles motivated by general relativity, namely locality and covariance.

One defines spaces $S(M, g)(x)$ of pointlike localized fields $\phi(M, g)(x)$ for each point x of the spacetime (M, g) and uses the master fields in order to ensure that the $\phi(M, g)(x)$ are the counterparts of the members of $S(0)$, the space of thermal observables at the origin of Minkowski spacetime. A state on the quantum field theory on the spacetime (M, g) is then called $S(M, g)(x)$-thermal if its expectation values on members of $S(M, g)(x)$ agree with the expectation values of some reference state on members of $S(0)$. One has thus utilized the master fields that define fields on all spacetimes in order to compare states in different spacetimes. Application of this seemingly abstract definition to specific models has yielded some interesting results within recent years; among them is a generalisation of the Unruh-temperature in de Sitter spacetime.

Of course, it is an important question whether there actually exist local thermal equilibrium states in the above sense. For quantum field theories in Minkowski space-time, Buchholz, Ojima and Roos showed that non-trivial (meaning not globally thermal equilibrium) $S(x)$-thermal equilibrium states exist at least for finite-dimensional $S(x)$. Later, Christoph Solveen showed this to be true for $S(O)$-thermal states for all compact O.

Just recently, using different methods, he could show assuming that $S(x)$ consists only of unbounded quadratic forms that in fact non-trivial $S(x)$-thermal states must exist on general curved spacetimes, if again $S(x)$ is finite-dimensional. The unboundedness of the quadratic forms is related to scaling behaviours of the local observables of the theory. Even more, he could prove, using continuity arguments that the above mentioned states are in fact $S(O)$-thermal for small neighbourhoods O of x.

Now Christoph sets out to find $S(O)$-thermal states on curved spacetimes. Moreover, current results by Stottmeister lead to the idea that intrinsic KMS-states contain information about condensation. This shall be studied using local thermal equilibrium states. Apart from that, Christoph hopes to be able to obtain results about thermal evolution properties by connecting microscopic information and global properties using local thermal equilibrium states in curved spacetime.

Written by Peter Schlicht.

Smooth Extension of Regular Cohomology

The research of Ingo Schröder

The problem Ingo Schröder wants to solve is to extend the complex cobordism on the infinite-dimensional complex projective space to smooth cobordism.

Why is complex cobordism so interesting?

The first example of a smooth cohomology theory that was studied was Deligne cohomology, the smooth equivalent to cohomology with integer coefficients. It became an object of interest because some higher invariants of manifolds naturally live in Deligne cohomology. Several models for Deligne cohomology were developed, for example Cheeger and Simons' "differential characters" in the 1980s. These models all highlight certain features of Deligne cohomology, but some of the other features are rather difficult to see in each model – for example, the "integration over the fiber" map or the product structure.

In order to better understand Deligne cohomology, several other smooth cohomology theories have been introduced in recent years, especially smooth K-theory and several versions of smooth bordism theories. The non-smooth versions of these cohomology theories are closely related by maps such as the Chern character, and hopefully these relations lift to the smooth extensions.

What is the complex cobordism?

Complex cobordism is a particular example of a generalized cohomology theory as they are studied by algebraic topologists. Basically, this means that we attach to manifold a family of groups numbered by non-negative integers, the so-called cohomology groups. Furthermore, a map between two manifolds induces maps between the associated groups indexed by the same integer but in the "wrong" way, which is why the prefix "co-" occurs; that is, if we imagine maps as arrows the direction gets reversed when we go from the world of manifolds to the world of groups by using cobordism. We require this passage from the manifold world to the world of groups to satisfy a rather long list of axioms. Complex cobordism is built on the very geometric notion of cobordism, which we explain now.

One calls two manifolds of the same dimension cobordant if there is a manifold with boundary, called the cobordism, in the next greater dimension whose boundary is the disjoint union of the two given manifolds. An easy example of two cobordant manifolds are two circles with the cobordism a soap film between them. A formal consequence of this definition is that a manifold who is itself a boundary is cobordant to the empty set.

Next, we consider a fixed manifold X of dimension n and define its pth cohomology group in terms of generators and relation between these. Generators are all possible "singular submanifolds" of X of codimension k, that is, pairs consisting of a manifold of dimension $n-k$ equipped with a map, which one might view as a generalized inclusion map, of this manifold to the bigger manifold X. In particular, each submanifold of codimension k is a generator. We consider two singular submanifolds of X, that is, pairs as above, to be equal or cobordant if the manifolds defining them are cobordant in such a way that the cobordism connecting them allows a map to X which restricts to the maps which are part of the given pairs. This gives the relations between the generators and thus defines the cobordism group of X. In this way, we get the unoriented cobordism ring. By varying the definitions, on can define the complex cobordism ring of a manifold X, which is roughly the group of cobordism classes of stably complex manifolds over X, where "stably complex" means that there is a complex linear structure on the stable normal bundle.

What is Ingo Schröder doing?

Ingo Schröder's work consists in constructing a geometric model of smooth complex bordism where representatives for classes in the smooth complex bordism group of a manifold X are explicitly given by a pair of a manifold M with a map from M to X and a differential form on X. The advantage of this geometric approach is that operations such as the product, the integration or the pullback can all explicitly be computed on the level of these representatives.

The other half of his work is to lift Chern classes to smooth complex cobordism. These lifted classes will not only be generalizations of ordinary characteristic classes, but they should also be related to the higher invariants mentioned above via the smooth versions of maps such as the Chern character. Furthermore, it is hoped that the smooth characteristic classes combined with his computation-friendly model of smooth complex bordism will make it easier to explicitly compute the higher invariants.

Written by Xiaowen Wu
with further simplifications and suggestions by Daniel Pape.

Geometrical Equivalence of Singularities and Associated Invariants

The research of Adrian Szawlowski

Consider two holomorphic maps as equivalent if they differ by composition on the right with a biholomorphic map that preserves a fixed volume or symplectic form. One can ask for invariants which can be associated to each such holomorphic map such that two maps are equivalent whenever their invariants coincide.

Let f and g be parallel holomorphic maps. We assume throughout that their domain is endowed with a volume form or a symplectic form, respectively, where in the latter case we assume, of course, that the dimension of the space is even. The two maps are called geometrically equivalent if one can be obtained from the other by precomposing with a biholomorphic map on the domain that preserves the forms. Moreover, geometrical equivalence may be studied locally, replacing biholomorphic maps by germs of such maps around zero. Some examples of local geometrical equivalences arise as follows. If f is a holomorphic submersion from a $2n$-dimensional symplectic space to the n-dimensional compact space such that certain Poisson brackets vanish, then there is an isomorphism acting between the germs around zero in the domain which preserves the symplectic form and pulls back the map to a fixed canonical one.

In the following, we turn to the problem of constructing invariants for geometrical equivalence. More precisely, we seek objects that can be assigned to each holomorphic map and which carry so much information that two maps are geometrical equivalent whenever these objects coincide. At first we consider the local case and assume our germ is represented by a holomorphic function on the m-dimensional complex space such that the origin is a non degenerate critical point, that is, the Jacobian vanishes at zero but the Hessian is non-degenerate there. Then, due to the so-called isochore Morse lemma, there are biholmorphic maps in the domain and codomain which conjugate our function to

$$x_1^2 + \ldots + x_m^2.$$

Moreover, the isomorphism in the image space is uniquely determined when tangent to the identity. Hence an invariant whose equality imply geometric equivalence is given by this function, which moreover fortunately can be computed in a geometrical way as follows. Each Morse map germ as above induces by restriction a fibration (called Milnor fibration) whose base is a small punctured disc around the origin in the plane.

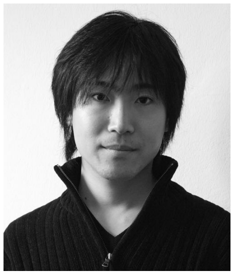
Now the associated invariant function can be computed by integration of a differential form on the typical fibre of this fibration over the canonical generator of the corresponding homology group which is infinite cyclic.

In his dissertation project Adrian is trying to construct similar invariants for non-isolated singularities and in the global case for polynomials on the whole n-dimensional complex space. For the latter, he uses a result that traces back to Thom and Verdier instead of the just locally defined Milnor fibrations. Accordingly, for each such polynomial exists a finite subset of the complex plane such the restriction of the polynomial to the inverse image of the complementary set is a locally trivial fibration. Moreover, this exceptional set contains the critical values of the polynomial. The local case inspires the conjecture that we get invariants for geometrical equivalence analogously by integration. But in order to study this idea it is necessary to understand the interplay between the topological and complex structure of the fibres.

Written by Nils Waterstraat.

New Techniques for the Classification of Nuclear C*-Algebras

The research of Mitsuharu Takeori



The doctoral student Mitsuharu Takeori is working on operator algebras. In this field, deeply connected with physics, the analysis of infinite-dimensional and non-commutative objects requires beautiful algebraic and topological tools. He is particularly interested in the classification of nuclear C*-algebras.

The field of operator algebras is the part of mathematics studying operators on infinite-dimensional spaces. It is a very important subject because it furnishes the structure of important physical theories like quantum theory. Mitsuharu Takeori is interested in C*-algebras, fundamental objects of the theory, which turn out to be a basic tool not only in physics, but also in many other areas of mathematics, like noncommutative geometry.

To understand the structure of C*-algebras, mathematicians are working on their classification, which is in general a very difficult problem. To classify means to find properties, or construct mathematical objects, through which the C*-algebras can be completely recognized. This is already possible for example for some important classes of C*-algebras like the class of *approximately finite-dimensional* C*-algebras, which were completely classified in terms of their K-theory.

Now the researchers of the field aim to classify more general classes of C*-algebras in terms of their K-theories. Mitsuharu Takeori works toward the classification of nuclear C*-algebras. He explains why these algebras are important: first, because of their relation to matrix algebras, given by the property of the approximation of the identity through completely positive maps to matrix algebras (this approximation property is equivalent to the uniqueness of tensor products); moreover, *nuclear* C*-algebras play an essential role in quantum physics, especially quantum statistical physics.

He is following a precise direction, recently introduced, toward the classification: this is the study of the so-called *C*-algebra bundles*. A natural step of thinking of a C*-algebra as a non-commutative analogue of the space of continuous functions is to try to view a C*-algebra as a set of sections of some kind of bundle. The motivation comes indeed from the commutative case, where any algebra $C_0(X)$ is the family of sections of a trivial bundle over X, naturally isomorphic to the topological space of primitive ideals.

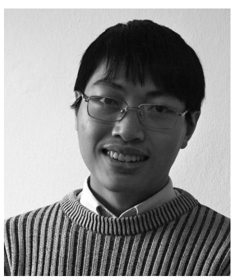
A C*-algebra bundle is by definition a C*-algebra together with a continuous map from its space of primitive ideal to a topological space X. This means that the space X, in its topology, encodes much information on the C*-algebra. For example, when the space is Hausdorff, the C*-algebra is really an algebra of continuous sections vanishing at infinity. On the other side, there are many important examples of C*-algebras which come with a canonical bundle structure. At the moment Mitsuharu Takeori is investigating the C*-bundles where X is a space with a finite number of points. This case is interesting because it corresponds to algebras with finitely many ideals, and still contains problems to be clarified.

If the space X gets more complicated, new techniques will be needed to describe the bundles. Mitsuharu Takeori's goal is to define and analyze the classes of nice C*-algebra bundles, which are expected to be classified by their K-theoretic data.

Written by Sara Azzali.

Microlocal Study of a Class of Degenerate Pseudodifferential Operators

The research of Nguyen Nhu Thang



Consider a differential operator on a manifold with boundary. There is a deep theory describing the elliptic case. Thang's project is to investigate the strictly hyperbolic case.

For an elliptic operator there is an "almost" inverse operator (a parametrix) of the same type. This leads to Boutet de Monvel's calculus. Briefly, an elliptic boundary value problem can be viewed as an element of a certain algebra defined by Boutet de Monvel. Hyperbolic operators are not so well-understood. Assume that a strictly hyperbolic pseudo-differential operator acts on a Sobolev space. The "inverse" operator is a degenerate pseudo-differential operator, which is a generalization of a pseudo-differential operator.

Further, for a strictly hyperbolic Cauchy problem there is a notion of Lagrangian submanifolds corresponding to the manifold, its boundary and the hyperbolic operator. These Lagrangian submanifolds and their intersections are important for developing a symbol calculus in the hyperbolic case. Thang's aim is to better understand possible arrangements of these Lagrangian submanifolds and then describe the symbol calculus for the inhomogeneous Cauchy problem in the corresponding cases. As a corollary, the Cauchy problem can be seen as an element in a special algebra.

Written by Dzmitry Dudko.

Infinite-Dimensional Index Theorems

The research of Nils Waterstraat

Nils Waterstraat's research is in functional analysis. In particular, he studies infinite-dimensional generalizations of the Morse Index Theorem.

The topology of a Riemannian manifold can be studied by classical Morse theory, which investigates Morse functions on such manifolds. A typical example of a Morse function is the height function on a torus which measures the height over some plane on which the torus sits. In this example we have four critical points (the two inner points and the two boundary points) and they contribute to the topology of our manifold. To make this precise one defines the notion of the so-called Morse index at each critical point: it is the maximal dimension of a subspace of the tangent space on which the Hessian form is negative definite with respect to the chosen Riemannian metric.

Classical Morse theory applies to finite-dimensional manifolds. Yet there are interesting examples in the infinite-dimensional case, too, for example the well-known energy functional which is defined on the infinite-dimensional space of paths in a Riemannian manifold. This functional is obtained by assigning to a path the integral over the square of the magnitude of its velocity vector. As is well known from physics this is proportional to the kinetic energy, which explains its name. What makes this functional so interesting is the fact that its critical points are exactly the geodesics in the Riemannian manifold.

Now the famous Morse Index Theorem states that the index of a geodesic can be described by counting Jacobi fields along the geodesic. These are solutions to a certain second order linear differential equation and can be thought of as tangents to a curve (that is, a family) of geodesics.

It has then become important for physicists to find a generalization of this theorem to Lorentz manifolds. Here it was shown that the index theorem cannot be rescued for space-like geodesics. Therefore, a further generalization to arbitrary semi-Riemannian manifolds was not easily conceivable. Yet a paper by Pejsachowicz, Musso and Portaluri from 2005 shows such a result on the functional analytic level. As a generalization of the Morse index they take the so-called spectral flow. Roughly speaking, it counts how many eigenvalues change their sign along a path of self-adjoint operators with discrete spectrum. For the analogue of the second ingredient in the index theorem, namely the contribution coming from the Jacobi fields, there are two possible choices: the conjugate index (using the Brouwer mapping degree) and the Maslov index. With these changes, the index theorem is still valid.

Nils used ideas of Pejsachowicz to find another proof of this theorem. And he further generalized the theorem: instead of defining a spectral flow along a path of self-adjoint operators, he defines it for a family of operators indexed by an arbitrary locally compact space. Now the spectral flow is no longer a number, but a K-theory class in the odd topological K-theory of the space that indexes the family.

Written by Adrian Szawlowski.

Higher Structures in Differential Geometry

The research of Chenchang Zhu

This is an introduction to Chenchang Zhu's research interests, her current and previous work and an answer to why it is difficult but also interesting.

Research Interests. Chenchang Zhu is working on higher structures in differential geometry. Her research interests are symplectic geometry, contact geometry, Poisson geometry, stacks, gerbes, Lie algebroids, Lie groupoids, simplicial manifolds and elliptic gamma functions.

Work and Previous Work. Currently, her main work is about Lie groupoids and Lie algebroids. Integrating Lie algebroids is a long-standing problem: unlike for finite-dimensional Lie algebras, which always have an associated Lie group, a Lie algebroid does not always come from a associated Lie groupoid. Non-integrability also shows up in the case of infinite-dimensional Lie algebras. In her dissertation, Chenchang Zhu found that a Lie algebroid can, nevertheless, always be integrated into an étale stack with a groupoid structure, which she calls a Weinstein groupoid. The converse is true as well; hence the Lie algebroid version of the one-to-one correspondence between Lie algebras and Lie groups is fully established. Applying the above to Jacobi manifolds, she proved that the integrating objects of Jacobi manifolds are contact Weinstein groupoids. She also determined when a Jacobi manifold can be integrated by a contact Lie groupoid. Her work, some done with colleagues, also contains some generalizations of Lie groupoids, establishes a functor to extend local Lie groupoids to Lie 2-groupoids, and studies the semi-direct product of a Lie algebra with a representation up to homotopy and provides various examples.

Why is it difficult? One of the difficulties is to find appropriate generalisations of smooth manifolds, such as differentiable stacks. But if there is a stacky Lie groupoids, how can one go back to the initial problem? Some of the difficulties are solved by previous work of other people, such as how to write down the partial differential equations for Lie algebroids or how to tell whether the solution is a manifold or not.

Why is it interesting? Higher Lie groupoids and higher Lie algebroids provide new notions of global and infinitesimal symmetries. People can use the higher category viewpoint to solve problems that cannot be solved in a traditional way. Such applications appear, for instance, in higher gauge theory.

Written by Jiguang Zheng
edited by Ralf Meyer.