A heuristic way of obtaining the Kerr metric

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An intuitive, straightforward way of finding the metric of a rotating black hole is presented, based on the algebra of differential forms. The representation obtained for the metric displays a simplicity which is not obvious in the usual Boyer–Lindquist coordinates.

I. INTRODUCTION

The formulation of general relativity by Albert Einstein in 1915 was one of the greatest advances of modern physics. It describes the dependence of the structure of space–time on the distribution of matter, and the converse effect of this space–time structure on matter distribution. Despite the overwhelming clarity of its foundation and the elegance of its basic equations, it has proved to be very difficult to find exact analytical solutions of the Einstein equations. Moreover, of all the exact solutions which are known, only a limited class seem to have a real physical meaning. Among them are the famous solutions of Schwarzschild and Kerr for black holes, and the Friedman solution for cosmology. Although the ''simple'' solution for a static, spherically symmetric black hole (static vacuum solution with spherical symmetry and central singularity) was found by Schwarzschild shortly after Einstein's publication of his equations, nearly 48 years were to elapse before Kerr[1] discovered the vacuum solution for the stationary axisymmetric rotating black hole.

Today, there exists a wealth of literature about solving the Einstein equations and about their solutions. Powerful new techniques were developed, like the spinor technique of Penrose and Newman,[2] or Bäcklund transformation techniques (for a review see, for example, Ref. 3). Despite their great success in treating the Einstein equations, these methods are technically complicated and are known mainly to the specialists working in the field. Taking into account the great physical importance of the Kerr solution, it is desirable to have a more straightforward way of finding it from the vacuum Einstein equations. Although a straightforward but nonetheless general way for finding the Kerr solution can be found in the classic work, *The Mathematical Theory of Black Holes*, by S. Chandrasekhar,[4] we will present here a more heuristic way of finding this solution, revealing its simplicity and elegance by using an alternative presentation of its metric. Technically, all calculations will be presented in the language of differential forms.

II. A HEURISTIC GUESS FOR THE METRIC OF A ROTATING BLACK HOLE

What we are looking for is the metric of a stationary, axially symmetric solution of the vacuum Einstein equations. The term ''stationary'' implies that there are no dependencies of the space–time structure on time t. Axial symmetry implies that there is no dependency on the coordinate of revolution in axially symmetric coordinates. In constructing a guess for the metric, we will first try to choose an appropriate coordinate presentation of the flat space–time metric. This is then modified by the introduction of additional unknown functions, whose form is subsequently determined by imposing the vacuum Einstein equations. The aim is to look for a simple guess for the metric of a rotating black hole. To do this, one can consider the known Schwarzschild solution of a static black hole and try to generalize it to a possible metric of a rotating black hole. The Schwarzschild space–time can be thought of as consisting of two-dimensional coordinate surfaces with the 2-metric of spheres (in Euclidean three-dimensional space), but coupled by an unusual radius function—the radial distance between two spheres with radial coordinates r_1 and r_2 is not r_2-r_1, but $\int_{r_1}^{r_2} dr\sqrt{g_{rr}}$, where g_{rr} is the rr component of the metric. Exactly this, together with a radius-dependent metric component g_{tt}, causes a nonvanishing curvature of the space–time. What could be a possible generalization of this space–time structure giving that of a rotating black hole? From a pure technical point of view, replacing the spheres by rotational ellipsoids is the simplest thing one can try. Both types of surfaces are described by simple second-order algebraic equations. Of course, there is no direct physical justification for such a guess, and the only way to prove its validity will be to solve the Einstein equations and to find a noncontradictory solution. However, simply replacing the spherical coordinate surfaces is not sufficient for a guess metric. For a rotating black hole one has to expect the occurrence of nonvanishing off-diagonal metric components (in a coordinate basis), coupling the angular coordinate with time.

Explicitly, for describing the spatial part of the space–time, we will use oblate spheroidal (orthogonal) coordinates ξ, θ, and ϕ.[5] Their coordinate surfaces can be expressed in Cartesian coordinates x,y,z by

$$\frac{x^2+y^2}{a^2\cosh^2\xi}+\frac{z^2}{a^2\sinh^2\xi}=1,$$
$$\frac{x^2+y^2}{a^2\cos^2\theta}-\frac{z^2}{a^2\sin^2\theta}=1, \tag{1}$$
$$y-\tan\phi x=0,$$

where a is a positive constant. The first of these equations describes ellipsoids of revolution ($\xi=$const.), the second, hyberboloids of revolution ($\theta=$const.), and the third, axial planes ($\phi=$const.). Some coordinate surfaces of this system are depicted in Fig. 1. The constant a is an arbitrary parameter, defining the location of the common foci of the ellipsoids.

By solving Eqs. (1) for x, y, and z, one finds

$$x=a\cosh\xi\cos\theta\cos\phi,$$
$$y=a\ \mathrm{cosht}\xi\cos\theta\sin\phi, \tag{2}$$
$$z=a\sinh\xi\sin\theta,$$

so that the Euclidean line element in these coordinates is given by

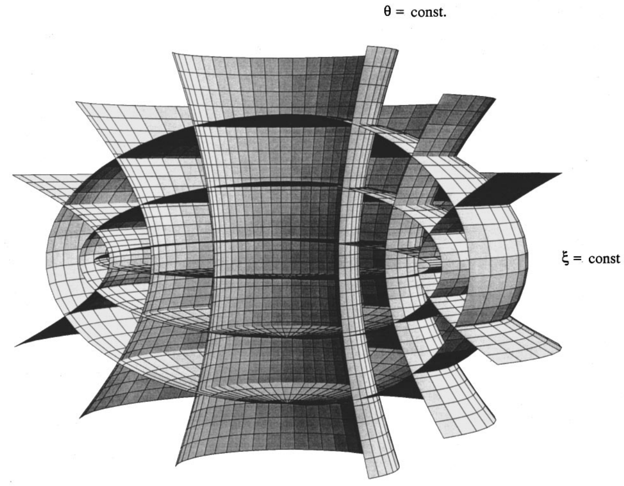

Fig. 1. Schematic representation of the oblate spheroidal coordinates. Surfaces of constant ξ (ellipsoids) and of constant θ (hyperboloids) are shown.

$$ds^2=dx^2+dy^2+dz^2=a^2(\sinh^2\xi+\sin^2\theta)(d\xi^2+d\theta^2)$$
$$+a^2\cosh^2\xi\cos^2\theta\, d\phi^2. \tag{3}$$

Next, the basis 1-forms in the above coordinates for the flat space–time have to be chosen. It is convenient for the subsequent calculations to choose them in such a way that the metric acquires the diagonal form $g_{\alpha\beta}=\eta_{\alpha\beta}$, where $\eta_{\alpha\beta}$ denotes the Minkowski flat space–time metric. Thus the basis one-forms of the flat space–time (zero curvature) in oblate spheroidal coordinates read

$$\tilde{\omega}^t=\mathbf{d}t,$$
$$\tilde{\omega}^\xi=a\Sigma\ \mathbf{d}\xi,$$
$$\tilde{\omega}^\theta=a\Sigma\ \mathbf{d}\theta, \tag{4}$$
$$\tilde{\omega}^\phi=a\cosh\xi\cos\theta\,\mathbf{d}\phi,$$

where the abbreviation $\Sigma=\sqrt{\sinh^2\xi+\sin^2\theta}$ was used.

As a first attempt, one would be inclined to use an ansatz similar to the Schwarzschild solution, i.e., one would multiply the basis 1-forms $\tilde{\omega}^t$ and $\tilde{\omega}^\xi$ of the flat space–time by two unknown functions, exp f and exp g. Obviously, this will only lead to the Schwarzschild solution itself, expressed in quite unfortunate coordinates. Recalling that one is looking for a rotating black hole solution, one could try to use the Lorentz transformed basis 1-forms $\cosh\beta\,\tilde{\omega}^t-\sinh\beta\,\tilde{\omega}^\phi$ and $\cosh\beta\,\tilde{\omega}^\phi-\sinh\beta\,\tilde{\omega}^t$ instead of $\tilde{\omega}^t$ and $\tilde{\omega}^\phi$, where $\beta=\beta(\xi,\theta)$ is a coordinate-dependent Lorentz transformation parameter. Then one arrives at the following set of basis 1-forms:

$$\tilde{\omega}^t=e^f[\cosh\beta\,\mathbf{d}t-\sinh\beta\,a\cosh\xi\cos\theta\,\mathbf{d}\phi],$$
$$\tilde{\omega}^\xi=e^g a\Sigma\ \mathbf{d}\xi,$$
$$\tilde{\omega}^\theta=a\Sigma\ \mathbf{d}\theta, \tag{5}$$
$$\tilde{\omega}^\phi=\cosh\beta\,a\cosh\xi\cos\theta\,\mathbf{d}\phi-\sinh\beta\,\mathbf{d}t,$$

which include the three unknown functions f, g, and β.

The use of such an ansatz for finding a stationary axially symmetric vacuum solution entails no essential loss of generality, as long as all three unknown functions are assumed to depend on both ξ and θ. If the coordinate surfaces defined by the basis 1-forms (5) are completely unrelated to the intrinsic character of the final solution, then this ansatz leads to over-complicated expressions for the Riemann curvature tensor and the Ricci tensor. It will therefore be assumed in the present paper that f and g depend *only* on ξ and not on θ. This singles out the ellipsoidal coordinate surfaces for the space–time structure being sought.

III. CALCULATING THE RICCI TENSOR

There follows a brief description of the calculation of the Riemann curvature and Ricci tensor in the language of differential forms, following mainly Ref. 6. The Einstein summation convention is used throughout.

First, we have to find the connection one-forms $\tilde{\omega}^{\alpha}_{\beta} = \Gamma^{\alpha}_{\beta\gamma}\tilde{\omega}^{\gamma}$, which are defined by the first Cartan relation:

$$\mathbf{d}\tilde{\omega}^{\alpha} = -\tilde{\omega}^{\alpha}_{\beta}\wedge\tilde{\omega}^{\beta} = -\Gamma^{\alpha}_{\beta\gamma}\tilde{\omega}^{\gamma}\wedge\tilde{\omega}^{\beta}. \tag{6}$$

Additionally, because the metric components in our basis are constant, the connection one-forms are antisymmetric [see Ref. 6, Eq. (14.31b)]:

$$\tilde{\omega}_{\alpha\beta} = -\tilde{\omega}_{\beta\alpha}. \tag{7}$$

Relation Eq. (6) is of no direct use for the calculation of the connection one-forms. But the new symbols $c^{\alpha}_{\beta\gamma}$ antisymmetric in β and γ defined by

$$\mathbf{d}\tilde{\omega}^{\alpha} = -\tfrac{1}{2}c_{\beta\gamma}{}^{\alpha}\tilde{\omega}^{\beta}\wedge\tilde{\omega}^{\gamma} \tag{8}$$

are easy to calculate by applying the exterior derivative to our basis one-forms. Comparing Eq. (8) with Eq. (6) one can see that $\Gamma_{\alpha\gamma\beta} - \Gamma_{\alpha\beta\gamma} = c_{\beta\gamma\alpha}$. By applying a cyclic permutation of the indices to $\Gamma_{\alpha\gamma\beta} - \Gamma_{\alpha\beta\gamma} = c_{\beta\gamma\alpha}$, and then adding/subtracting the resulting equations, one finds that

$$\tilde{\omega}^{\alpha}_{\beta} = g^{\alpha\lambda}\tilde{\omega}_{\lambda\beta} = \tfrac{1}{2}g^{\alpha\lambda}(c_{\lambda\beta\gamma} + c_{\lambda\gamma\beta} - c_{\beta\gamma\lambda})\tilde{\omega}^{\gamma}. \tag{9}$$

For our basis one-forms as defined by Eq. (5), the connection one-forms explicitly read

$$\begin{aligned}\tilde{\omega}^{\,t}_{\ \xi} = {} & \tilde{\omega}^{\,t}(a\Sigma e^{g})^{-1}[-\sinh^2\beta\tanh\xi + f'] - \tilde{\omega}^{\phi}(a\Sigma e^{g})^{-1}\\ & \times[\cosh f\cosh\beta\sinh\beta\tanh\xi + \sinh f\,\beta_{,\xi}],\end{aligned} \tag{10a}$$

$$\begin{aligned}\tilde{\omega}^{\,t}_{\ \theta} = {} & \tilde{\omega}^{\,t}(a\Sigma)^{-1}\sinh^2\beta\tan\theta + \tilde{\omega}^{\phi}(a\Sigma)^{-1}\\ & \times[\cosh f\cosh\beta\sinh\beta\tan\theta - \sinh f\,\beta_{,\theta}],\end{aligned} \tag{10b}$$

$$\begin{aligned}\tilde{\omega}^{\,t}_{\ \phi} = {} & \tilde{\omega}^{\xi}(a\Sigma e^{g})^{-1}[\sinh f\cosh\beta\sinh\beta\tanh\xi\\ & + \cosh f\,\beta_{,\xi}] + \tilde{\omega}^{\theta}(a\Sigma)^{-1}[-\sinh f\cosh\beta\\ & \times\sinh\beta\tan\theta + \cosh f\,\beta_{,\theta}],\end{aligned} \tag{10c}$$

$$\begin{aligned}\tilde{\omega}^{\xi}_{\theta} = {} & \tilde{\omega}^{\xi}(a\Sigma^{3})^{-1}\cos\theta\sin\theta\\ & - \tilde{\omega}^{\theta}(a\Sigma^{3}e^{g})^{-1}\cosh\xi\sinh\xi,\end{aligned} \tag{10d}$$

$$\begin{aligned}\tilde{\omega}^{\xi}_{\phi} = {} & -\tilde{\omega}^{\,t}(a\Sigma e^{g})^{-1}[\cosh f\cosh\beta\sinh\beta\tanh\xi\\ & + \sinh f\,\beta_{,\xi}] - \tilde{\omega}^{\phi}(a\Sigma e^{g})^{-1}\cosh^2\beta\tanh\xi,\end{aligned} \tag{10e}$$

$$\begin{aligned}\tilde{\omega}^{\theta}_{\phi} = {} & \tilde{\omega}^{\,t}(a\Sigma)^{-1}[\cosh f\cosh\beta\sinh\beta\tan\theta - \sinh f\,\beta_{,\theta}]\\ & + \tilde{\omega}^{\phi}(a\Sigma)^{-1}\cosh^2\beta\tan\theta.\end{aligned} \tag{10f}$$

Here, a prime denotes differentiation by ξ, and $\beta_{,\xi}$ and $\beta_{,\theta}$ denote the two partial derivatives of β.

Next, one calculates the Riemann curvature 2-form, which is defined by the second Cartan relation:

$$\tilde{R}^{\alpha}_{\beta} \equiv \tfrac{1}{2}R^{\alpha}_{\beta\gamma\delta}\tilde{\omega}^{\gamma}\wedge\tilde{\omega}^{\delta} = \mathbf{d}\tilde{\omega}^{\alpha}_{\beta} + \tilde{\omega}^{\alpha}_{\gamma}\wedge\tilde{\omega}^{\gamma}_{\beta}. \tag{11}$$

The $R^{\alpha}_{\beta\gamma\delta}$ are the components of the Riemann tensor. Inserting Eqs. (10) into the last equation, one obtains the following nonvanishing and distinct components of the Riemann tensor:

$$\begin{aligned}R^{t}_{\xi t\xi} = {} & [a^2e^{2g}\Sigma^2]^{-1}\{(\sinh^2 f - \sinh 2f)(\cosh^2\beta\sinh^2\beta\tanh^2\xi + \beta^2_{,\xi}) + (\sinh 2f - 4\sinh^2 f)\cosh\beta\sinh\beta\\ & \times\tanh\xi\,\beta_{,\xi} - \Sigma^{-2}\sinh^2\beta\sin^2\theta\,(e^{2g}-1) + \Sigma^{-2}\cosh\xi\sinh\xi\, f' - f'^2 + \sinh^2\beta\tanh\xi\,(2f' - g') + f'g' - f''\},\end{aligned} \tag{12a}$$

$$\begin{aligned}R^{t}_{\xi t\theta} = {} & [2a^2e^{g}\Sigma^2]^{-1}\{2(\sinh 2f - \sinh^2 f)[\cosh^2\beta\sinh^2\beta\tanh\xi\tan\theta - \beta_{,\xi}\beta_{,\theta}] + (\sinh 2f - 4\sinh^2 f)\\ & \times\cosh\beta\sinh\beta[\tanh\xi\,\beta_{,\theta} - \tan\theta\,\beta_{,\xi}] + 2\Sigma^{-2}\sinh^2\beta(\cosh\xi\sinh\xi\tan\theta - \tanh\xi\cos\theta\sin\theta)\\ & - \sinh^2\beta(1-\sinh^2\beta)\tanh\xi\tan\theta + 2(\Sigma^{-2}\cos\theta\sin\theta - \sinh^2\beta\tan\theta)f'\},\end{aligned} \tag{12b}$$

$$\begin{aligned}R^{t}_{\xi\xi\phi} = {} & [a^2e^{2g}\Sigma^2]^{-1}\{\cosh f[\Sigma^{-2}\cosh\beta\sinh\beta\sin^2\theta\,(e^{2g}-1) - \cosh\beta\sinh\beta\tanh\xi\,(f'-g') - 2f'\beta_{,\xi}]\\ & + \sinh f[\cosh\beta\sinh\beta(\cosh^2\beta + \sinh^2\beta)\tanh^2\xi - 2\cosh\beta\sinh\beta\tanh\xi\, f' - \Sigma^{-2}e^{2g}\cos\theta\sin\theta\,\beta_{,\theta}\\ & + (\Sigma^{-2}\cosh\xi\sinh\xi - \tanh\xi - f' + g')\beta_{,\xi} - \beta_{,\xi\xi}]\},\end{aligned} \tag{12c}$$

$$\begin{aligned}R^{t}_{\xi\theta\phi} = {} & [a^2e^{g}\Sigma^2]^{-1}\{\cosh f[\Sigma^{-2}\cosh\beta\sinh\beta(\cos\theta\sin\theta\tanh\xi - \cosh\xi\sinh\xi\tan\theta) + \cosh\beta\sinh\beta\\ & \times\tan\theta(\tanh\xi + f') - f'\beta_{,\theta}] - \sinh f[\cosh\beta\sinh\beta(\cosh^2\beta + \sinh^2\beta)\tanh\xi\tan\theta - (\cosh\beta\sinh\beta\tan\theta - \beta_{,\theta})f'\\ & - (\Sigma^{-2}\cosh\xi\sinh\xi + \sinh^2\beta\tanh\xi)\beta_{,\theta} - (\Sigma^{-2}\cosh\theta\sin\theta + \cosh^2\beta\tan\theta)\beta_{,\xi} + \beta_{,\xi\theta}]\},\end{aligned} \tag{12d}$$

$$\begin{aligned}R^{t}_{\theta t\theta} = {} & [a^2\Sigma^2]^{-1}\{(\sinh^2 f - \sinh 2f)[\cosh^2\beta\sinh^2\beta\tan^2\theta + \beta^2_{,\theta}] + (4\sinh^2 f - \sinh 2f)\cosh\beta\sinh\beta\tan\theta\,\beta_{,\theta}\\ & + \Sigma^{-2}\sinh^2\beta\sinh^2\xi(e^{-2g}-1) - e^{-2g}\Sigma^{-2}\cosh\xi\sinh\xi\, f'\},\end{aligned} \tag{12e}$$

$$R^{t}_{\theta\xi\phi}=[a^2e^{g}\Sigma^2]^{-1}\{\cosh f[\Sigma^{-2}\cosh\beta\sinh\beta(\cos\theta\sin\theta\tanh\xi-\cosh\xi\sinh\xi\tan\theta)+\cosh\beta\sinh\beta\tanh\xi\times\tan\theta-f'\beta_{,\theta}]-\sinh f[\cosh\beta\sinh\beta(\cosh^2\beta+\sinh^2\beta)\tanh\xi\tan\theta-\cosh\beta\sinh\beta\tanh\xi f'-(\Sigma^{-2}\cosh\xi\sinh\xi-\cosh^2\beta\tanh\xi)\beta_{,\theta}-(\Sigma^{-2}\cos\theta\sin\theta-\sinh^2\beta\tan\theta)\beta_{,\xi}+\beta_{,\xi\theta}]\},\tag{12f}$$

$$R^{t}_{\theta\theta\phi}=[a^2\Sigma^2]^{-1}\{\Sigma^{-2}\cosh f\cosh\beta\sinh\beta\sinh^2\xi(1-e^{-2g})+\sinh f[\cosh\beta\sinh\beta(\cosh^2\beta+\sinh^2\beta)\tan^2\theta+(\Sigma^{-2}\cos\theta\sin\theta+\tan\theta)\beta_{,\theta}-e^{-2g}\Sigma^{-2}\cosh\xi\sinh\xi\,\beta_{,\xi}-\beta_{,\theta\theta}]\},\tag{12g}$$

$$R^{t}_{\phi^t\phi}=[a^2e^{2g}\Sigma^2]^{-1}\{\sinh 2f\cosh\beta\sinh\beta(e^{2g}\tan\theta\,\beta_{,\theta}-\tanh\xi\,\beta_{,\xi})-\sinh^2 f[\cosh^2\beta\sinh^2\beta(e^{2g}\tan^2\theta+\tanh^2\xi)+e^{2g}\beta^2_{,\theta}+\beta^2_{,\xi}]-\cosh^2\beta\tanh\xi f'\},\tag{12h}$$

$$R^{t}_{\phi\xi\theta}=[a^2e^{g}\Sigma^2]^{-1}\{\sinh f[\beta_{,\theta}f'-(\cosh^2\beta+\sinh^2\beta)(\tanh\xi\,\beta_{,\theta}+\tan\theta\,\beta_{,\xi})]-\cosh f\cosh\beta\sinh\beta\tan\theta f'\},\tag{12i}$$

$$R^{\xi}_{\theta\xi\theta}=[a^2e^{2g}\Sigma^6]^{-1}\{(\sinh^2\xi\cos^2\theta-\cosh^2\xi\sin^2\theta)(1-e^{2g})+\Sigma^2\cosh\xi\sinh\xi\, g'\},\tag{12j}$$

$$R^{\xi}_{\phi\xi\phi}=[a^2e^{2g}\Sigma^2]^{-1}\{(\sinh 2f+\sinh^2 f)(\cosh^2\beta\sinh^2\beta\tanh^2\xi+\beta^2_{,\xi})+(4\sinh^2 f+\sinh 2f)\times\cosh\beta\sinh\beta\tanh\xi\,\beta_{,\xi}+\Sigma^{-2}\cosh^2\beta\sin^2\theta(e^{2g}-1)+\cosh^2\beta\tanh\xi\, g'\},\tag{12k}$$

$$R^{\xi}_{\phi\theta\phi}=[2a^2e^{g}\Sigma^2]^{-1}\{(4\sinh^2 f+\sinh 2f)\cosh\beta\sinh\beta(\tanh\xi\,\beta_{,\theta}-\tan\theta\,\beta_{,\xi})-2(\sinh^2 f+2\sinh 2f)\times[\cosh^2\beta\sinh^2\beta\tanh\xi\tan\theta-\beta_{,\xi}\beta_{,\theta}]+2\Sigma^{-2}\cosh^2\beta(\cos\theta\sin\theta\tanh\xi-\cosh\xi\sinh\xi\tan\theta)+\cosh^2\beta(\cosh^2\beta+1)\tanh\xi\tan\theta\},\tag{12l}$$

$$R^{\theta}_{\phi\theta\phi}=[a^2\Sigma^2]^{-1}\{(\sinh^2 f+\sinh 2f)[\cosh^2\beta\sinh^2\beta\tan^2\theta+\beta^2_{,\theta}]-(4\sinh^2 f+\sinh 2f)\cosh\beta\sinh\beta\tan\theta\,\beta_{,\theta}+\Sigma^{-2}\cosh^2\beta\sinh^2\xi(1-e^{-2g})\}.\tag{12m}$$

All other nonvanishing components of the Riemann tensor can be found by employing the symmetry properties of $R^{\alpha}_{\beta\gamma\delta}=g^{\alpha\mu}R_{\mu\beta\gamma\delta}$:

$$R_{\alpha\beta\gamma\delta}=R_{\gamma\delta\alpha\beta}=-R_{\beta\alpha\gamma\delta}=-R_{\alpha\beta\delta\gamma}.\tag{13}$$

Finally, by contracting the Riemann tensor, one finds the components of the Ricci tensor:

$$R^{\alpha}_{\beta}=R^{\gamma\alpha}{}_{\gamma\beta}.\tag{14}$$

For our space–time metric, the Ricci tensor components are:

$$R^{t}_{t}=(a\Sigma e^{g})^{-2}\{(1-e^{2g})\sinh^2\beta+\sinh^2\beta\tanh\xi\,(f'-g')-\tanh\xi f'-f'^2+f'g'-f''-\sinh 2f[\cosh^2\beta\times\sinh^2\beta(e^{2g}\tan^2\theta+\tanh^2\xi)+e^{2g}\beta^2_{,\theta}+\beta^2_{,\xi}]+4\sinh^2 f\cosh\beta\sinh\beta(e^{2g}\tan\theta\,\beta_{,\theta}-\tanh\xi\,\beta_{,\xi})\},\tag{15a}$$

$$R^{t}_{\phi}=(a\Sigma e^{g})^{-2}\{\cosh f[\cosh\beta\sinh\beta(1-e^{2g}+f'-g')+2f'\beta_{,\xi}]+\sinh f[\cosh\beta\sinh\beta(2\tanh\xi f'-(e^{2g}\tan^2\theta+\tanh^2\xi)\cosh 2\beta)+\beta_{,\xi}(f'-g')+e^{2g}(\beta_{,\theta\theta}-\tan\theta\,\beta_{,\theta})+\beta_{,\xi\xi}+\tanh\xi\,\beta_{,\xi}]\},\tag{15b}$$

$$R^{\xi}_{\xi}=(a\Sigma e^{g})^{-2}\{2\sinh^2\beta\tanh\xi f'-f'^2+\tanh\xi\, g'+f'g'-f''-1+\Sigma^{-2}[e^{2g}(2\sin^2\theta-\cos^2\theta)-\cosh^2\xi+\cosh\xi\sinh\xi\,(f'+g')]+2\Sigma^{-4}[e^{2g}\cos^2\theta\sin^2\theta+\cosh^2\xi\sinh^2\xi]+2\sinh f\cosh\beta\sinh\beta\tanh\xi\,\beta_{,\xi}+2\sinh^2 f[\cosh^2\beta\sinh^2\beta\tanh^2\xi+\beta^2_{,\xi}]\},\tag{15c}$$

$$R^{\xi}_{\theta}=(a\Sigma)^{-2}e^{-g}\{(\cosh^2\beta\sinh^2\beta+1)\tan\theta\tanh\xi-\sinh^2\beta\tan\theta f'-\beta_{,\xi}\beta_{,\theta}+\Sigma^{-2}[\cos\theta\sin\theta(\tanh\xi+f')-\cosh\xi\sinh\xi\tan\theta]+2\sinh^2 f[\beta_{,\xi}\beta_{,\theta}-\cosh^2\beta\sinh^2\beta\tan\theta\tanh\xi]+\sinh 2f\cosh\beta\sinh\beta[\tanh\xi\,\beta_{,\theta}-\tan\theta\,\beta_{,\xi}]\},\tag{15d}$$

$$R^{\theta}_{\theta}=(a\Sigma)^{-2}\{\Sigma^{-2}\sinh^2\xi\,(1-e^{-2g})-\Sigma^{-2}\cosh\xi\sinh\xi\,(f'-g')+\Sigma^{-4}[\cosh^2\xi\sin^2\theta-\sinh^2\xi\cos^2\theta](1-e^{-2g})+2\sinh^2 f[\cosh^2\beta\sinh^2\beta\tan^2\theta+\beta^2_{,\theta}]-2\sinh 2f\cosh\beta\sinh\beta\tan\theta\,\beta_{,\theta}\},\tag{15e}$$

$$R^{\phi}_{\phi}=(a\Sigma e^{g})^{-2}\{\cosh^2\beta[e^{2g}-1-\tanh\xi\,(f'-g')]+4\sinh^2 f\cosh\beta\sinh\beta[\tanh\xi\,\beta_{,\xi}-e^{2g}\tan\theta\,\beta_{,\theta}]+\sinh 2f[\cosh^2\beta\sinh^2\beta(\tanh^2\xi+e^{2g}\tan^2\theta)+\beta^2_{,\xi}+e^{2g}\beta^2_{,\theta}]\}.\tag{15f}$$

Although the recipe for calculating the Ricci tensor is relatively simple, the resulting algebraic calculations may be time-consuming, and it is recommended that one uses symbolic programs for the calculations, like MAPLE or MATHEMATICA. All calculations in the present paper were done with the help of MATHEMATICA.[7]

IV. FINDING A SOLUTION OF THE VACUUM EINSTEIN EQUATIONS

For the vacuum, the right-hand side of the Einstein equations,

$$R^{\alpha}_{\beta}-(1/2)\,\delta^{\alpha}_{\beta}R=8\pi T^{\alpha}_{\beta}, \tag{16}$$

vanishes, and the equations reduce to $R^{\alpha}_{\beta}=0$. When considering Eqs. (15), it seems to be impossible to find a straightforward solution for the unknown functions f, g, and β. But one can use the fact that the unknown functions f and g do not depend on the coordinate θ. The idea is to consider the equations $R^{\alpha}_{\beta}=0$ in the limits $\theta\to 0$ and $\theta\to\pi/2$. When taking the latter limit one has to be careful because of the divergence of tan θ. Performing a series expansion of the Ricci tensor around $\theta=\pi/2$ reveals that the coefficient of the term with the highest divergence, $\sim(\pi/2-\theta)^{-2}$, has the form

$$\frac{\cosh\,\beta\,\sinh\,\beta}{2a^2\,\cosh^2\,\xi}\begin{pmatrix} -\sinh 2f\sinh 2\beta & 0 & 0 & -2\sinh f\,\cosh 2\beta \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 2\sinh^2 f\,\sinh 2\beta & 0 \\ 2\sinh f\,\cosh 2\beta & 0 & 0 & \sinh 2f\sinh 2\beta \end{pmatrix}. \tag{17}$$

Since β is assumed to be a smooth function, this implies that

$$\beta(\xi,\pi/2)=0,$$

$$\beta_{,\theta}(\xi,\pi/2)=0, \tag{18}$$

and that any derivative of β with respect to ξ on the symmetry axis equals zero. This also cancels automatically the divergent term proportional to $(\pi/2-\theta)^{-1}$. Assuming also that β is everywhere differentiable and symmetric with respect to the equatorial plane, one finds the additional constraint

$$\beta_{,\theta}(\xi,0)=0. \tag{19}$$

For the limits $\theta\to 0$ and $\theta\to\pi/2$, the derivatives of the unknown functions are distributed within the Ricci tensor as

$$\lim_{\theta\to 0} R^{\mu}_{\nu}\sim\begin{bmatrix} \{f',f'',g',\beta_{,\xi}\} & \varnothing & \varnothing & \begin{Bmatrix} f',g',\beta_{,\theta\theta} \\ \beta_{,\xi},\beta_{,\xi\xi} \end{Bmatrix} \\ \varnothing & \{f',f'',g',\beta_{,\xi}\} & \varnothing & \varnothing \\ \varnothing & \varnothing & \{f',g'\} & \varnothing \\ \begin{Bmatrix} f',g',\beta_{,\theta\theta} \\ \beta_{,\xi},\beta_{,\xi\xi} \end{Bmatrix} & \varnothing & \varnothing & \{f',g',\beta_{,\xi}\} \end{bmatrix} \tag{20}$$

and as

$$\lim_{\theta\to\pi/2} R^{\mu}_{\nu}\sim\begin{bmatrix} \{f',f'',g',\beta_{,\theta}\} & \varnothing & \varnothing & \{\beta_{,\theta\theta}\} \\ \varnothing & \{f',f'',g'\} & \varnothing & \varnothing \\ \varnothing & \varnothing & \{f',g',\beta_{,\theta}\} & \varnothing \\ \{\beta_{,\theta\theta}\} & \varnothing & \varnothing & \{f',g',\beta_{,\theta}\} \end{bmatrix}. \tag{21}$$

As a first step, one can try to exclude the two unknowns f'' and $\beta_{,\theta}$ in the equations $\lim_{\theta\to\pi/2} R^{\alpha}_{\beta}=0$. By examining the coefficients of these two functions in the components of the Ricci tensor, one finds that $\lim_{\theta\to\pi/2}(R^{t}_{t}-R^{\xi}_{\xi}+R^{\theta}_{\theta})$ will eliminate both terms:

$$\lim_{\theta\to\pi/2}(R^{t}_{t}-R^{\xi}_{\xi}+R^{\theta}_{\theta})=\frac{(1-\sinh^2\,\xi)[1-e^{2g}]-\cosh\,\xi\,\sinh\,\xi\,(3f'+g')}{a^2e^{2g}\,\cosh^4\,\xi}. \tag{22}$$

Moreover, the limit $\lim_{\theta\to 0}\,R^{\theta}_{\theta}$ also does not contain both terms:

$$\lim_{\theta\to 0}\,R^{\theta}_{\theta}=\frac{(1-\sinh^2\,\xi)[1-e^{2g}]-\cosh\,\xi\,\sinh\,\xi\,(f'-g')}{a^2e^{2g}\,\sinh^4\,\xi}. \tag{23}$$

Subtracting the numerators of Eqs. (22) and (23) yields $f'(\xi)=-g'(\xi)$. Taking into account that both f and g tend to zero for $\xi\to\infty$ (flat Minkowski space–time at infinity), one has $f(\xi)=-g(\xi)$. Substitution of this relation into Eq. (23) and integrating the latter leads to

$$\exp[f(\xi)]=\exp[-g(\xi)]=\sqrt{1-A\,\frac{\sinh\,\xi}{\cosh^2\,\xi}}, \tag{24}$$

with A as an integration constant.

It remains to find a solution for $\beta(\xi,\theta)$. One can seek it by close examination of the equation $R^{\theta}_{\theta}=0$, which results in

$$[4\sin^2\theta\cosh^2\xi+4\Sigma^4\beta_{,\theta}\tan\theta\cosh\beta\sinh\beta]A\,\frac{\sinh\xi}{\cosh^2\xi}$$
$$-[4\sin^2\theta\cosh^2\xi-\Sigma^4(\beta_{,\theta}-\tan\theta\cosh\beta\sinh\beta)^2]$$
$$\times\left(A\,\frac{\sinh\xi}{\cosh^2\xi}\right)^2=0. \qquad (25)$$

This equation still looks quite complicated, but one can satisfy Eq. (25) by making the two square brackets vanish separately. This leads to the solution $\cosh\beta=\Sigma^{-1}\cosh\xi$ and hence $\sinh\beta=\Sigma^{-1}\cos\theta$, where the integration constant is determined by taking into account $\lim_{\theta\to\pi/2}\beta=0$. Since, for any fixed ξ, this solution obeys the nonlinear ordinary differential equation (ODE) Eq. (25) with the boundary condition $\lim_{\theta\to\pi/2}\beta=0$, it is the only solution by the uniqueness theorem for ODEs. By directly inserting the resulting solutions of f, g, and β into the Ricci tensor, it is straightforward to prove that they really constitute a solution of the vacuum Einstein equations.

V. CONCLUSION

Collecting all results of the last section, the resulting metric finally has the form

$$ds^2=-\left[1-A\,\frac{\sinh\xi}{\cosh^2\xi}\right]\left(\frac{\cosh\xi}{\Sigma}\,dt-\frac{\cos\theta}{\Sigma}\,a\cosh\xi\cos\theta\,d\phi\right)^2$$
$$+\left[1-A\,\frac{\sinh\xi}{\cosh^2\xi}\right]^{-1}a^2\Sigma^2d\xi^2+a^2\Sigma^2d\theta^2$$
$$+\left(\frac{\cosh\xi}{\Sigma}\,a\cosh\xi\cos\theta\,d\phi-\frac{\cos\theta}{\Sigma}\,dt\right)^2. \qquad (26)$$

By changing the variables ξ to $r=a\sinh\xi$ and θ to $\pi/2-\theta$, one finds the equivalent form

$$ds^2=-\frac{\Delta}{\rho^2}[dt-a\sin^2\theta\,d\phi]^2+\frac{\sin^2\theta}{\rho^2}[(r^2+\rho^2)d\phi$$
$$-a\,dt]^2+\frac{\rho^2}{\Delta}dr^2+\rho^2d\theta^2, \qquad (27)$$

where the abbreviations $\Delta=r^2-2Mr+a^2$, $\rho^2=r^2+a^2\cos^2\theta$, and $2M\equiv Aa$ were used. Equation (27) is the standard representation of the Kerr metric in Boyer–Lindquist coordinates for a rotating black hole with mass M and angular momentum $S=aM$ [see Eqs. (33.2–4) in Ref. 6]. But now, the original simplicity of the metric as displayed in Eqs. (5) is no longer obvious.

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BLACK HOLES

The absence of any sharp spike, or cusp, of light on sub-arc-second scales bolsters the idea that M15 probably does not have a black hole in its midst... . For me, this rather mundane development was welcome news. While teaching college astronomy courses over the years, I had resisted the temptation to endow the heart of virtually every poorly understood object in the Universe with a black hole. The ''bandwagon'' appeal among astronomers who would have black holes lurking in darkened nooks and crannies practically everywhere—in the centers of galaxies, star clusters, exploding stars, even at the core of our Sun—was unconvincing to me, especially since there is no unambiguous evidence that even one such black hole actually exists anywhere. They probably do, but they could just as well be figments of our imagination.

Eric J. Chaisson, *The Hubble Wars* (HarperCollins, New York, 1994), p. 299.