Numerical methods for time harmonic locally perturbed periodic media

Part 3

Sonia Fliss
(Ecole Nationale Supérieure de Techniques Avancées)

Intensive Programme / Summer School „Periodic Structures in Applied Mathematics“
Göttingen, August 18 – 31, 2013

This project has been funded with support from the European Commission. This publication [communication] reflects the views only of the author, and the Commission cannot be held responsible for any use which may be made of the information contained therein.

Grant Agreement Reference Number: D-2012-ERA/MOBIP-3-29749-1-6

Numerical methods for time harmonic scalar wave equation in locally perturbed periodic media - Part 3

Sonia Fliss

POEMS (UMR 7231 CNRS-INRIA-ENSTA)

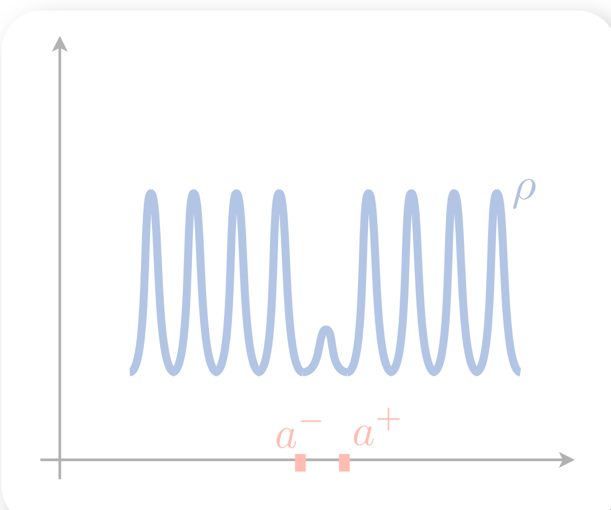
Mainly based on joint works with Julien Coatleven, Patrick Joly and Jing-Rebecca Li

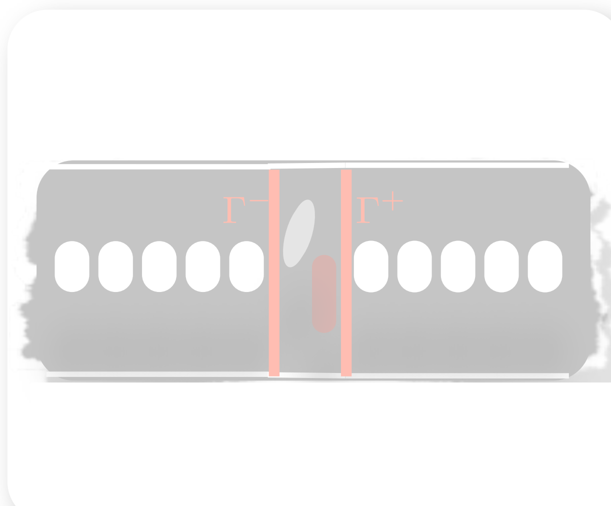
1D Case

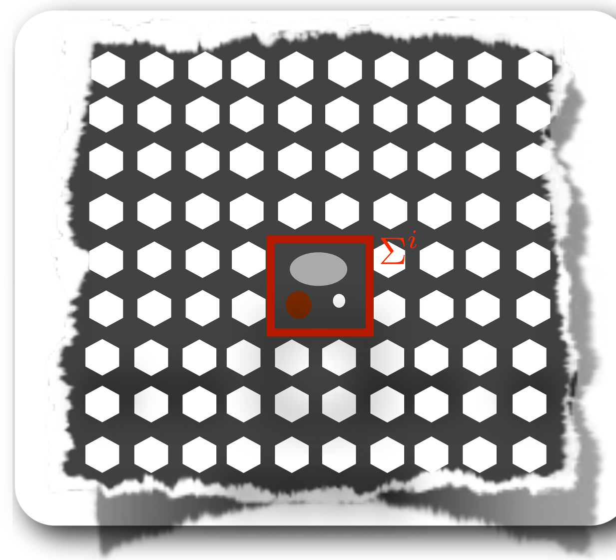
Waveguide case

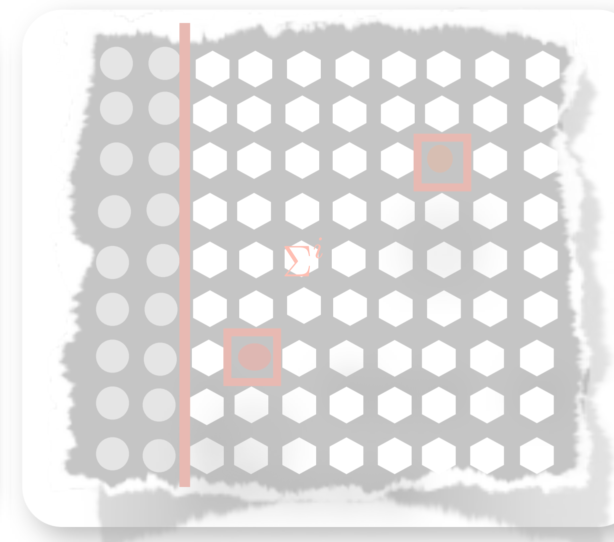
2D case
Simple scattering

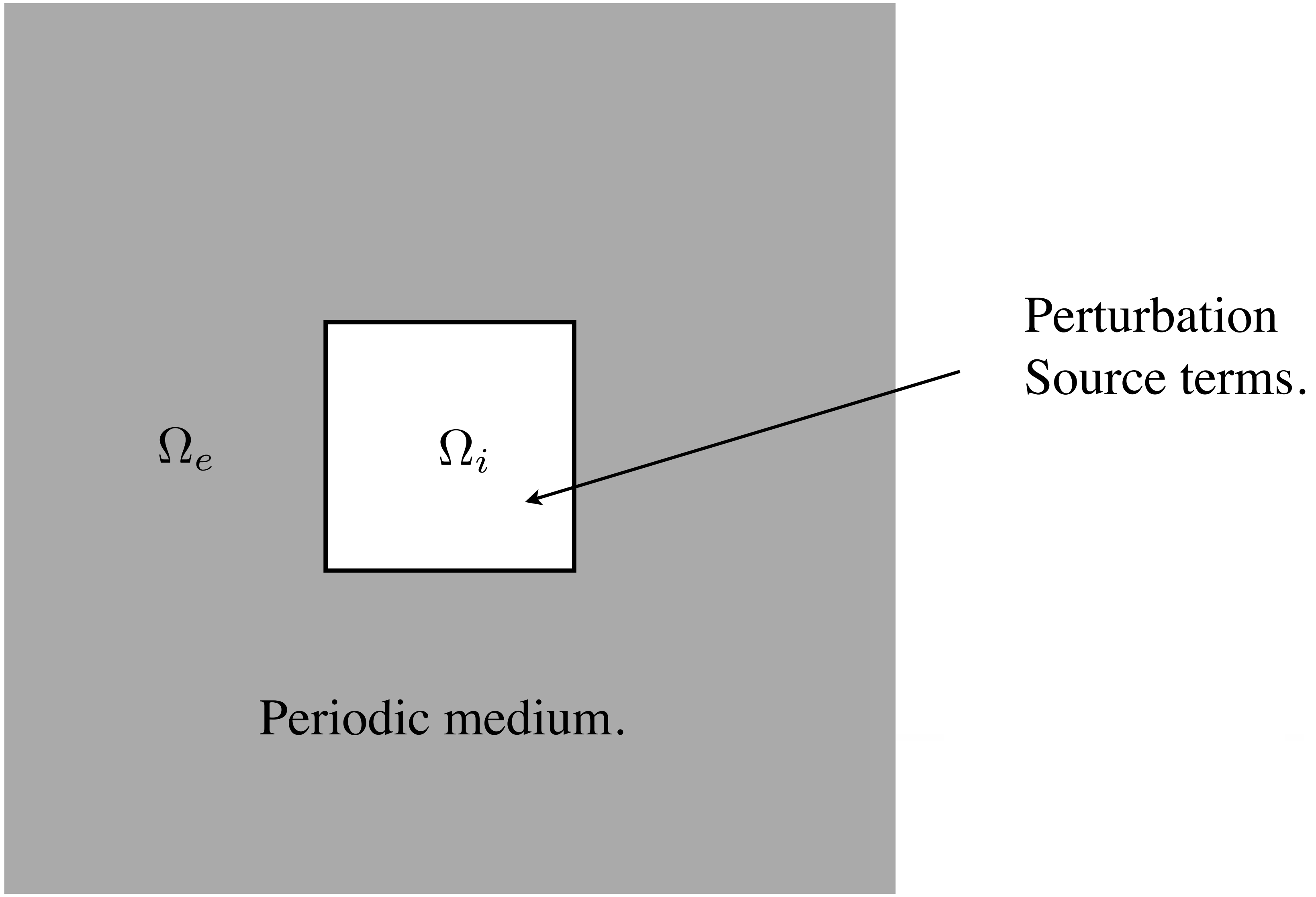
2D case
Multiple scattering



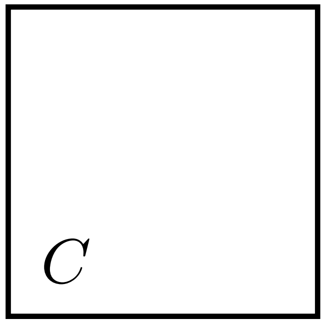




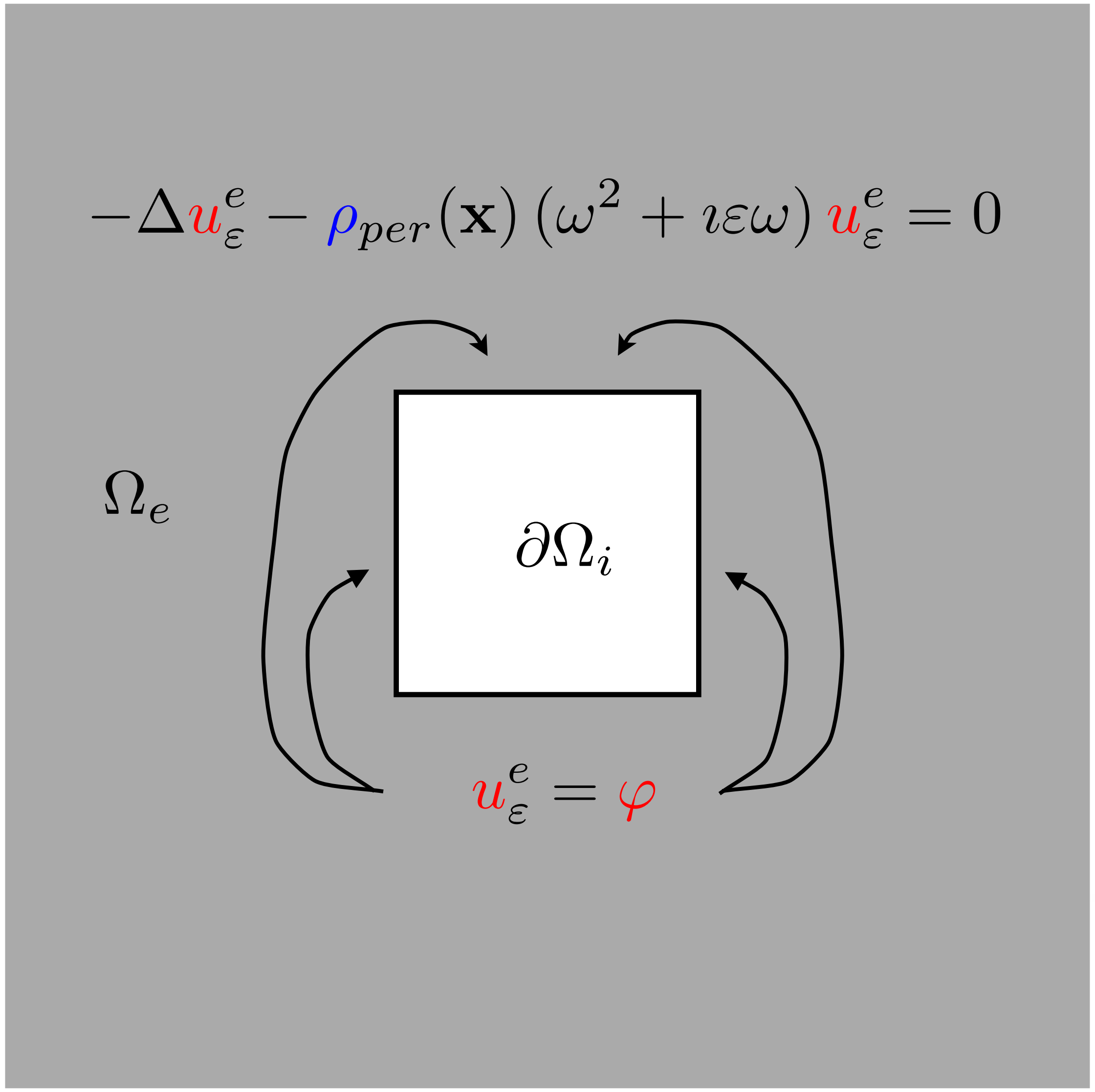




Assumption : The reference cell C is the unit square.



$u^e_\varepsilon = u^e_\varepsilon(\varphi)$

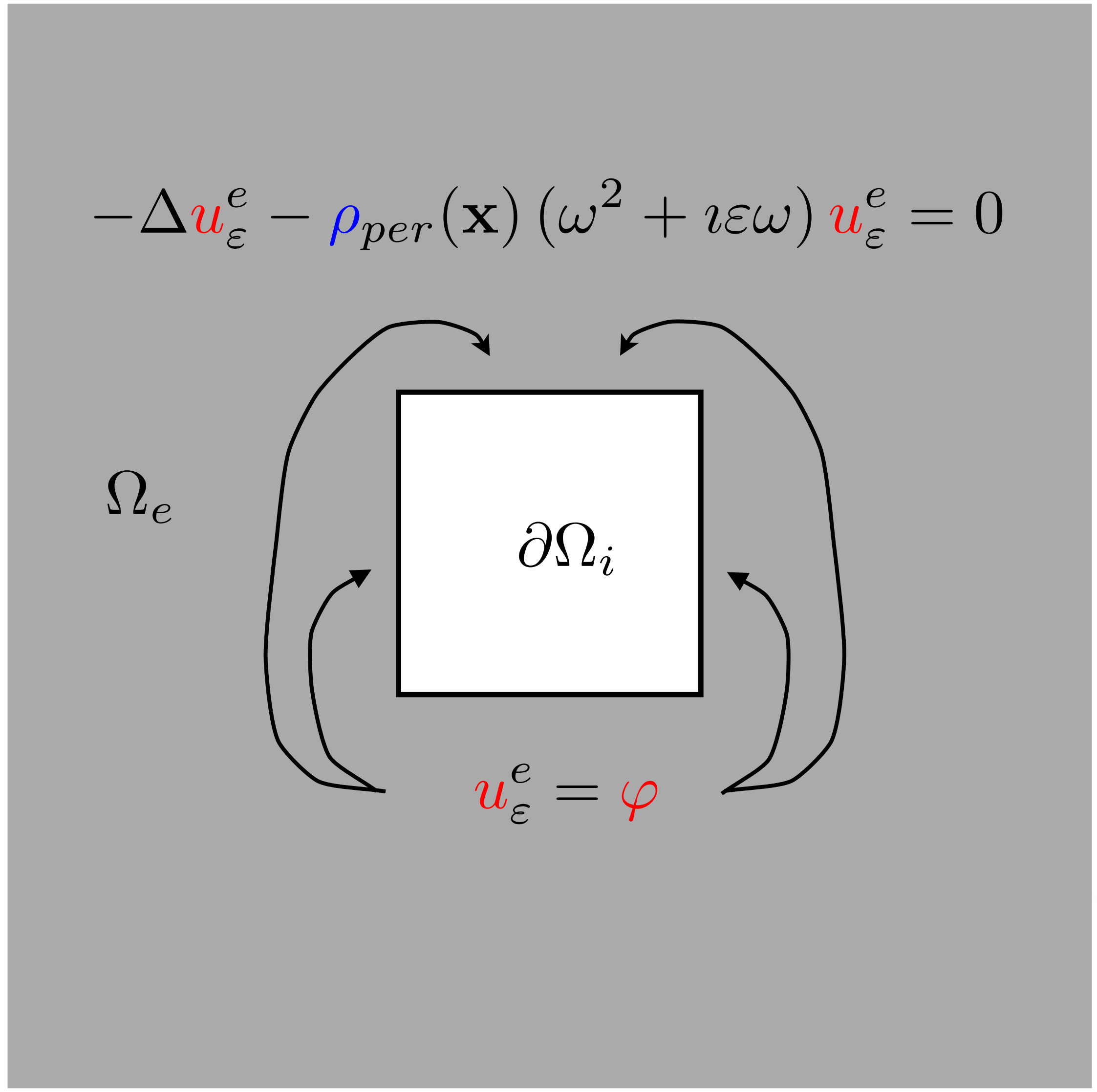


Assumption : The reference cell C is the unit square. Ω_i is a square made of $N \times N$ cells.

For the simplicity of the presentation, we shall assume that $N = 1$.

The case of a localized defect in the plane

$$u_\varepsilon^e = u_\varepsilon^e(\varphi)$$

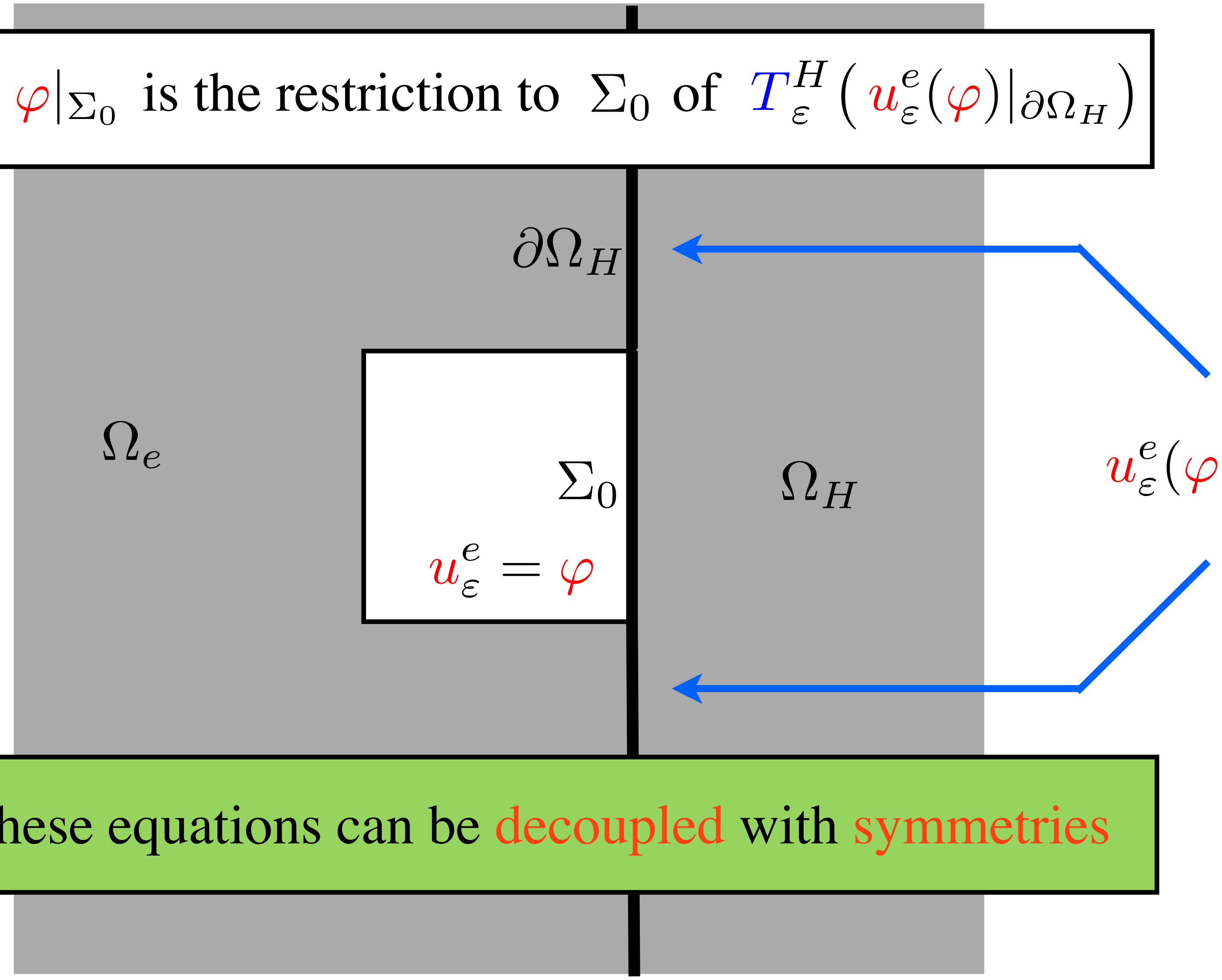


$$T_\varepsilon\,\varphi := -\,\partial_n u_\varepsilon^e(\varphi)|_{\partial\Omega_i} \qquad T_\varepsilon : H^{\frac{1}{2}}(\partial\Omega_i) \longrightarrow H^{-\frac{1}{2}}(\partial\Omega_i)$$

The case of a localized defect in the plane

$T_\varepsilon\,\varphi|_{\Sigma_0}$ is the restriction to Σ_0 of $T_\varepsilon^H\left(u_\varepsilon^e(\varphi)|_{\partial\Omega_H}\right)$

$u_\varepsilon^e = u_\varepsilon^e(\varphi)$



$u_\varepsilon^e(\varphi)|_{\partial\Omega_H}$?

These equations can be decoupled with symmetries

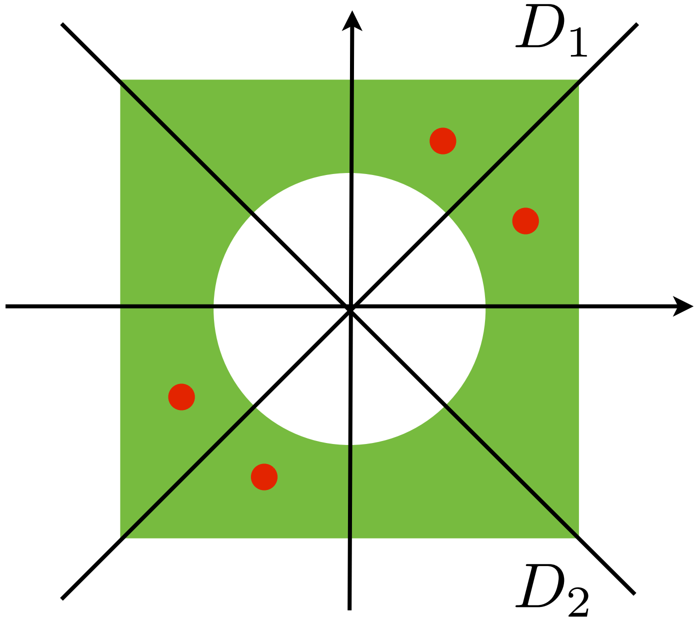
Can we build an operator : $\varphi \longrightarrow u_\varepsilon^e(\varphi)|_{\partial\Omega_H}$?

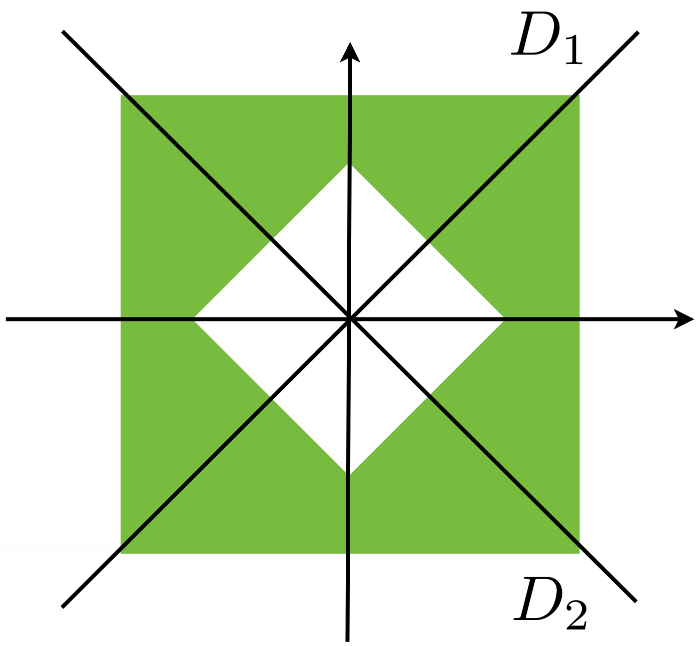
we need to build four such extension operators E_ε^j

These operators satisfy coupled equations

Double symmetry and consequences

A subset $\mathcal{O}$ of $\mathbb{R}^2$, with barycenter at the origin, will be called doubly symmetric if it is invariant by the two symmetries S_1 and S_2 with respect to $y = x$ and $y = -x$





A function $f : \mathcal{O} \longrightarrow \mathbb{C}$ is doubly symmetric if and only if

$$\forall\, \mathbf{x} \in \mathcal{O}, \quad f(S_1\mathbf{x}) = f(S_2\mathbf{x}) = f(\mathbf{x})$$

Assumption : the function $\rho_{per}(\mathbf{x}) : C \longrightarrow \mathbb{R}^+$ is doubly symmetric.

Symmetries and consequences

Double symmetry and consequences

A vector space $V(\mathcal{O})$ of functions from $\mathcal{O}$ into $\mathbb{C}$ can be decomposed as :

$$V(\mathcal{O}) = V_{ss}(\mathcal{O}) \oplus V_{sa}(\mathcal{O}) \oplus V_{as}(\mathcal{O}) \oplus V_{ss}(\mathcal{O})$$

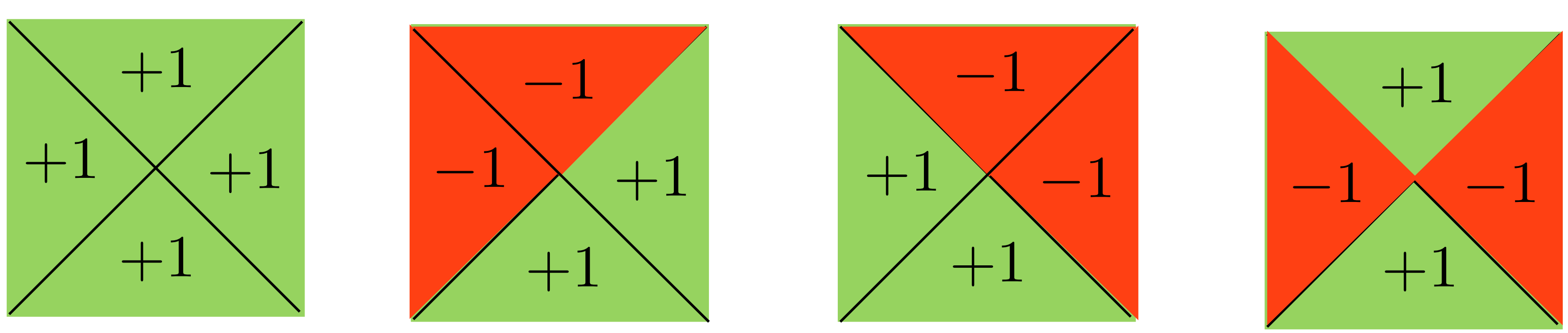
where by definition

$$V_{ss}(\mathcal{O}) = \left\{ v \in V(\mathcal{O}) \,/\, v(\mathbf{x}) = v(S_1\mathbf{x}) = v(S_2\mathbf{x}) \right\}$$

$$V_{sa}(\mathcal{O}) = \left\{ v \in V(\mathcal{O}) \,/\, v(\mathbf{x}) = v(S_1\mathbf{x}) = -v(S_2\mathbf{x}) \right\}$$

$$V_{as}(\mathcal{O}) = \left\{ v \in V(\mathcal{O}) \,/\, v(\mathbf{x}) = -\,v(S_1\mathbf{x}) = v(S_2\mathbf{x}) \right\}$$

$$V_{aa}(\mathcal{O}) = \left\{ v \in V(\mathcal{O}) \,/\, v(\mathbf{x}) = -\,v(S_1\mathbf{x}) = -\,v(S_2\mathbf{x}) \right\}$$



Double symmetry and consequences

A vector space $V(\mathcal{O})$ of functions from $\mathcal{O}$ into $\mathbb{C}$ can be decomposed as :

$$V(\mathcal{O}) = V_{ss}(\mathcal{O}) \oplus V_{sa}(\mathcal{O}) \oplus V_{as}(\mathcal{O}) \oplus V_{ss}(\mathcal{O})$$

with projectors

$$\Pi_{ss}v(\mathbf{x}) = \frac{1}{4}\Big\{v(\mathbf{x}) + v(S_1\mathbf{x}) + v(S_2\mathbf{x}) + v(S_1S_2\mathbf{x})\Big\}$$

$$\Pi_{as}v(\mathbf{x}) = \frac{1}{4}\Big\{v(\mathbf{x}) - v(S_1\mathbf{x}) + v(S_2\mathbf{x}) - v(S_1S_2\mathbf{x})\Big\}$$

$$\Pi_{sa}v(\mathbf{x}) = \frac{1}{4}\Big\{v(\mathbf{x}) + v(S_1\mathbf{x}) - v(S_2\mathbf{x}) - v(S_1S_2\mathbf{x})\Big\}$$

$$\Pi_{aa}v(\mathbf{x}) = \frac{1}{4}\Big\{v(\mathbf{x}) - v(S_1\mathbf{x}) - v(S_2\mathbf{x}) + v(S_1S_2\mathbf{x})\Big\}$$

Double symmetry and consequences For any $(p,q) \in \{a,s\}^2$

$$\boxed{\varphi \in H^{\frac{1}{2}}_{pq}(\partial\Omega_i) \quad \Longrightarrow \quad u^e_\varepsilon(\varphi) \in H^1_{pq}(\Delta;\Omega_e) \quad \Longrightarrow \quad \partial_n u^e_\varepsilon(\varphi) \in H^{-\frac{1}{2}}_{pq}(\partial\Omega_i)}$$

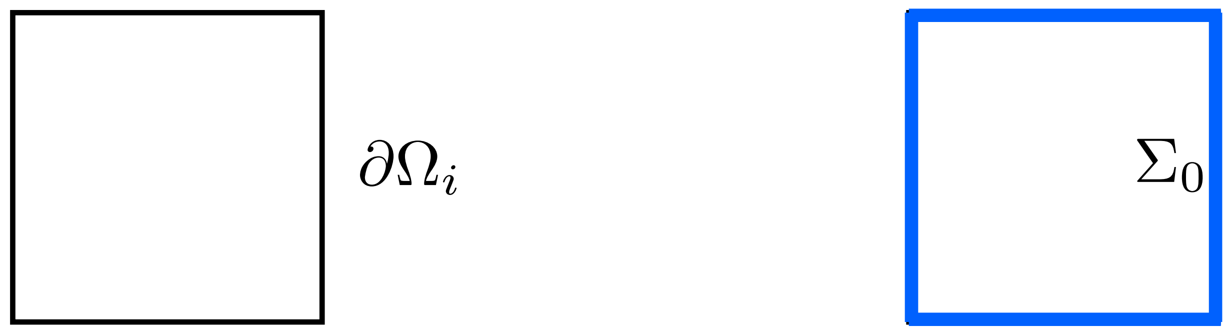
This property results from the fact that the Laplace operator and the multiplication by the doubly symmetric function ρ_{per} commute with the maps $u \longrightarrow u \circ S_j, \; j=1,2$

As a consequence T_ε maps $H^{\frac{1}{2}}_{pq}(\partial\Omega_i)$ into $H^{-\frac{1}{2}}_{pq}(\partial\Omega_i) \subset \left(H^{\frac{1}{2}}_{pq}(\partial\Omega_i)\right)'$

$$T_\varepsilon = \begin{array}{|c|c|c|c|} \hline T^{ss}_\varepsilon & 0 & 0 & 0 \\ \hline 0 & T^{sa}_\varepsilon & 0 & 0 \\ \hline 0 & 0 & T^{as}_\varepsilon & 0 \\ \hline 0 & 0 & 0 & T^{aa}_\varepsilon \\ \hline \end{array}$$

We are reduced to determine each T^{pq}_ε , $(p,q) \in \{a,s\}^2$. In the sequel $(p,q) = (s,s)$.

Factorization of the operator T_ε^{ss}



For $\varphi \in H_{ss}^{\frac{1}{2}}(\partial\Omega_i)$, knowing φ on $\partial\Omega_i$ is equivalent to knowing φ on Σ_0.

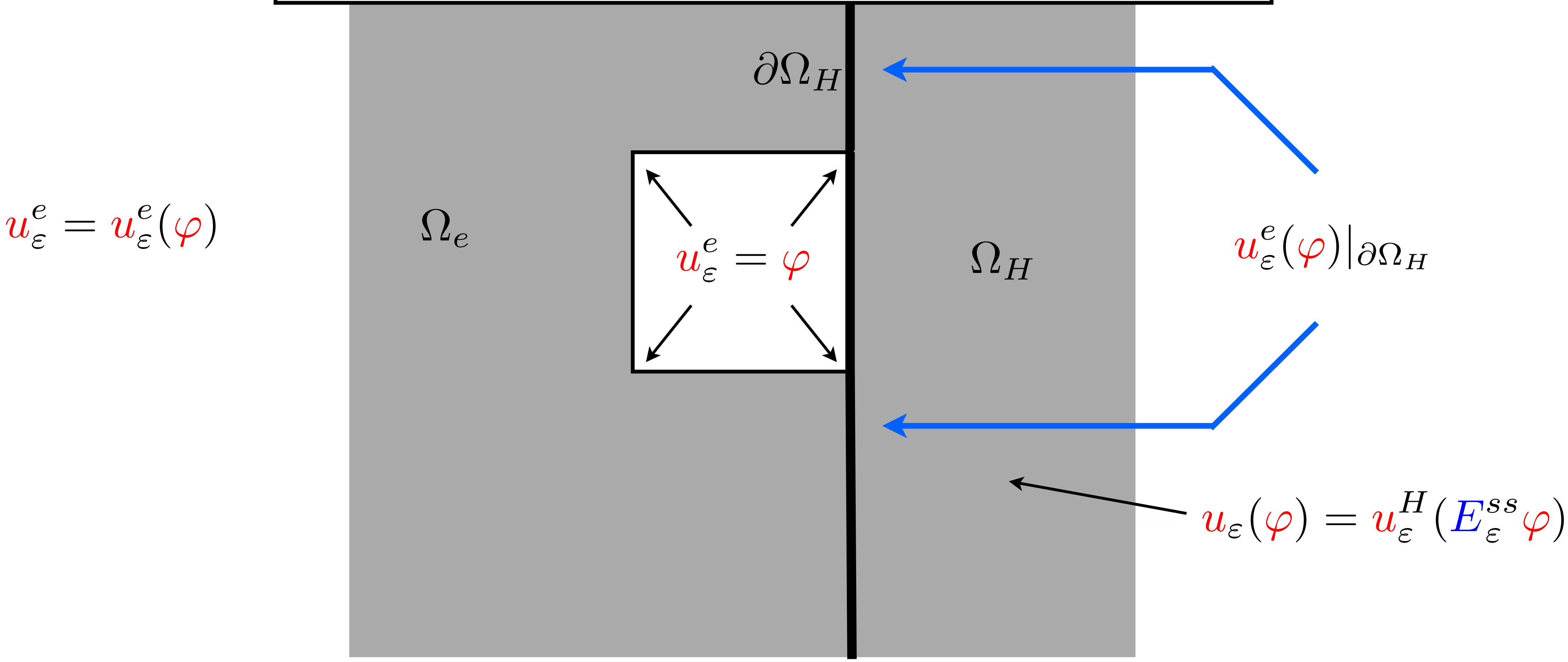
$H_{ss}^{\frac{1}{2}}(\partial\Omega_i)$ is isomorphic to $H^{\frac{1}{2}}(\partial\Sigma_0)$

$$H_{ss}^{\frac{1}{2}}(\partial\Omega_i) \xrightarrow{\chi} H^{\frac{1}{2}}(\partial\Sigma_0)$$

$$H_{ss}^{\frac{1}{2}}(\partial\Omega_i) \xleftarrow{\chi_{ss}^*} H^{\frac{1}{2}}(\partial\Sigma_0)$$

The case of a localized defect in the plane

$$\varphi \in H^{\frac{1}{2}}_{ss}(\partial\Omega_i) \longrightarrow u^e_\varepsilon(\varphi) \in H^1_{ss}(\Omega_e)$$

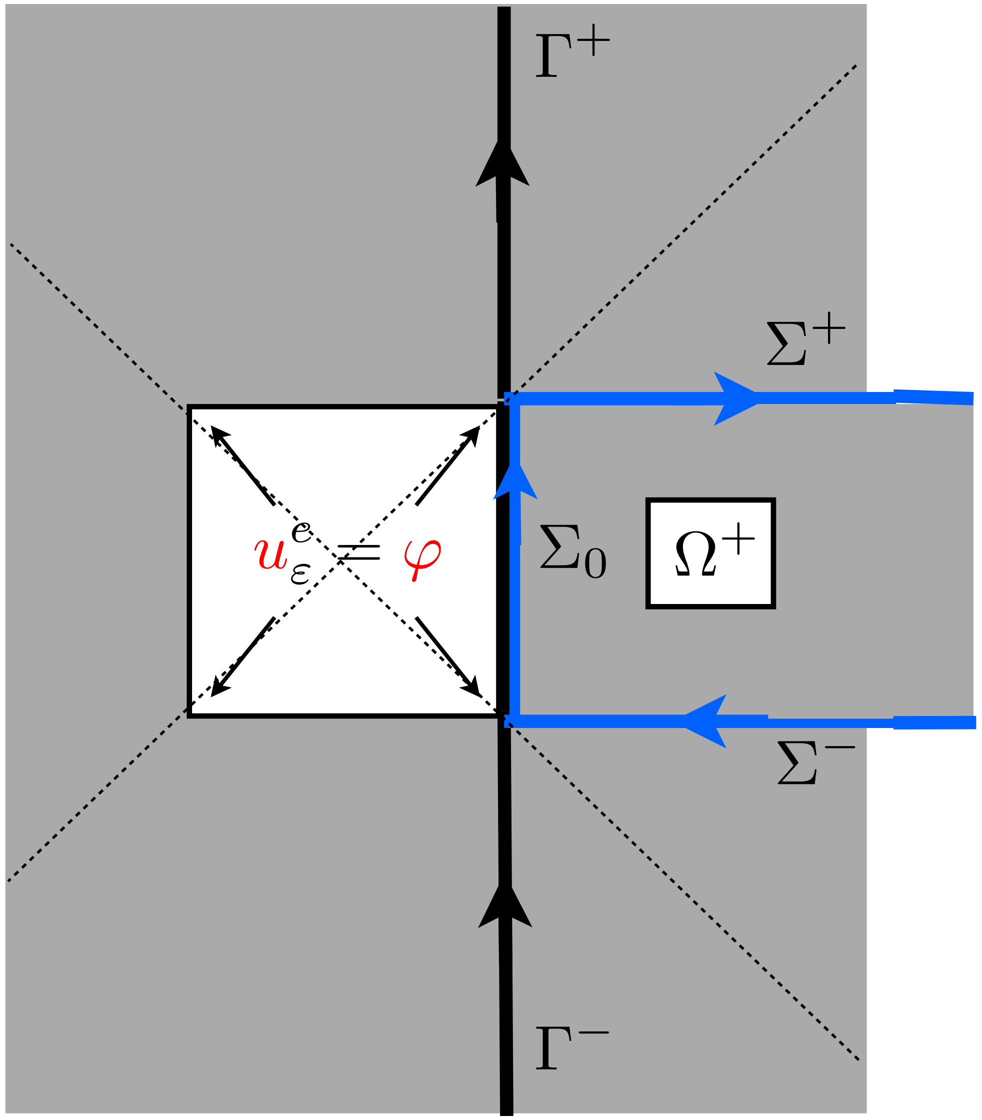


$$E^{ss}_\varepsilon : \varphi \in H^{\frac{1}{2}}_{ss}(\partial\Omega_i) \longrightarrow u^e_\varepsilon(\varphi)|_{\partial\Omega_H} \in H^{\frac{1}{2}}(\partial\Omega_H)$$

$$T^{ss}_\varepsilon = \chi^*_{ss} \circ \chi\, T^H_\varepsilon \circ E^{ss}_\varepsilon$$

For other symmetries $\quad T^{pq}_\varepsilon = \chi^*_{pq} \circ \chi\, T^H_\varepsilon \circ E^{pq}_\varepsilon \qquad (p,q) \in \{a,s\}^2$

A method for the construction of E_ε^{ss}



$$E_\varepsilon^{ss} \in \mathcal{L}_0 \subset \mathcal{L}\Big(H_{ss}^{\frac{1}{2}}(\partial\Omega_i), H^{\frac{1}{2}}(\partial\Omega_H)\Big)$$

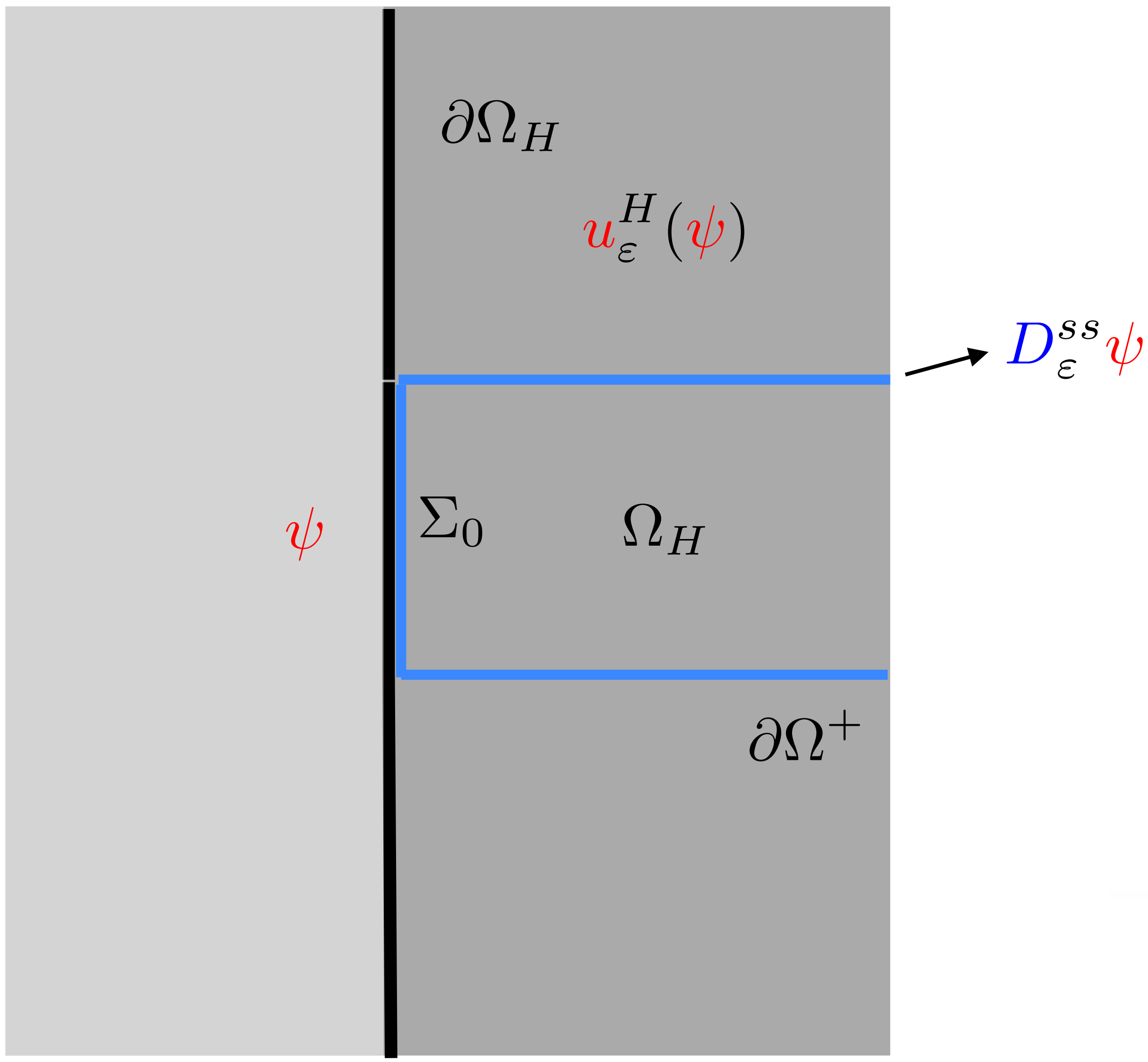
$$L \in \mathcal{L}_0 \iff \forall\varphi,\ \mathcal{L}\varphi = \varphi \text{ on } \Sigma_0$$

$$\partial\Omega^H = \Gamma^- \cup \Sigma_0 \cup \Gamma^+$$
$$|||$$
$$\partial\Omega^+ = \Sigma^- \cup \Sigma_0 \cup \Sigma^+$$

$$\Gamma^+ \equiv \Sigma^+ \qquad \Gamma^- \equiv \Sigma^-$$

$$u_\varepsilon(\varphi) = u_\varepsilon^H(E_\varepsilon^{ss}\varphi)$$

A method for the construction of E_ε^{ss}

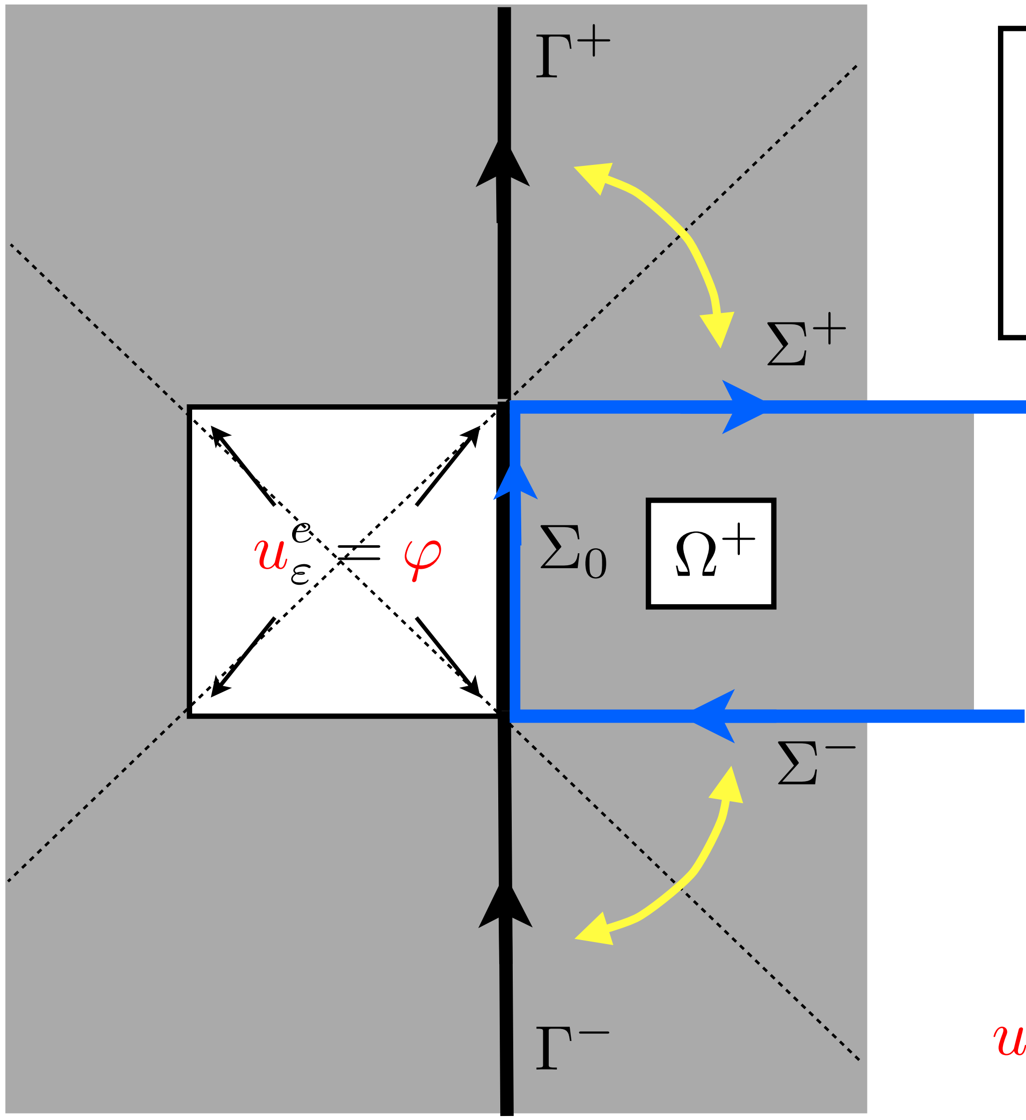


Let us introduce a new operator

$$\boxed{\begin{array}{c} D_\varepsilon^{ss} \in \mathcal{L}\big(H^{\frac{1}{2}}_{ss}(\partial\Omega_H), H^{\frac{1}{2}}(\partial\Omega_H)\big) \\[2ex] D_\varepsilon^{ss}\psi \equiv u_\varepsilon^H(\psi)|_{\partial\Omega^+} \end{array}}$$

$$D_\varepsilon^{ss}\psi|_{\Sigma_0} = \psi|_{\Sigma_0}$$

$$D_\varepsilon^{ss} \in \mathcal{L}_0$$



$$E_\varepsilon^{ss} \in \mathcal{L}_0 \subset \mathcal{L}\big(H_{ss}^{\frac{1}{2}}(\partial\Omega_i), H^{\frac{1}{2}}(\partial\Omega_H)\big)$$

$$L \in \mathcal{L}_0 \iff \forall\varphi,\ \mathcal{L}\varphi = \varphi \text{ on } \Sigma_0$$

$$\partial\Omega^+ = \Sigma^- \cup \Sigma_0 \cup \Sigma^+$$
$$|||$$
$$\partial\Omega^H = \Gamma^- \cup \Sigma_0 \cup \Gamma^+$$

$$\Gamma^+ \equiv \Sigma^+ \qquad \Gamma^- \equiv \Sigma^-$$

$$\underset{\downarrow \atop E_\varepsilon^{ss}\varphi}{u_\varepsilon^e(\varphi)|_{\Gamma^+}} \equiv \underset{\downarrow \atop u_\varepsilon^H(E_\varepsilon^{ss}\varphi)}{u_\varepsilon^e(\varphi)|_{\Sigma^+}} \qquad \underset{\downarrow \atop E_\varepsilon^{ss}\varphi}{u_\varepsilon^e(\varphi)|_{\Gamma^-}} \equiv \underset{\downarrow \atop u_\varepsilon^H(E_\varepsilon^{ss}\varphi)}{u_\varepsilon^e(\varphi)|_{\Sigma^-}}$$

$$u_\varepsilon(\varphi) = u_\varepsilon^H(E_\varepsilon^{ss}\varphi)$$

$$D_\varepsilon^{ss}\big(E_\varepsilon^{ss}\varphi\big) = E_\varepsilon^{ss}\varphi$$

Theorem : Characterization of E_ε^{ss}

The operator E_ϵ^{ss} is characterized as the unique solution of the problem

Find $E \in \mathcal{L}_0$ satisfying $D_\varepsilon^{ss} \circ E = E$

where $\mathcal{L}_0 = \left\{ L \in \mathcal{L}\left(H_{ss}^{\frac{1}{2}}(\partial\Omega_i), H^{\frac{1}{2}}(\partial\Omega_H)\right) \;/\; \forall\, \varphi,\ L\varphi = \varphi \text{ on } \Sigma_0 \right\}$ $L|_{\Sigma_0} = Id$

We have to solve a linear equation with an affine constrain

Proof : we only need to prove the uniqueness of the solution

$$D_\varepsilon^{ss} \circ E = E \quad \text{and} \quad E \in \overrightarrow{\mathcal{L}_0} \implies E = 0$$

$$\overrightarrow{\mathcal{L}_0} = \left\{ L \in \mathcal{L}\left(H_{ss}^{\frac{1}{2}}(\partial\Omega_i), H^{\frac{1}{2}}(\partial\Omega_H)\right) \;/\; \forall\, \varphi,\ L\varphi = 0 \text{ on } \Sigma_0 \right\}$$

$$\forall \varphi \in H^{\frac{1}{2}}(\partial\Omega_i), \quad D_\varepsilon^{ss} \circ E\varphi = E\varphi \quad \text{and} \quad E \in \mathcal{L}_0 \implies E\varphi = 0$$

$$\forall\, \psi \in H^{\frac{1}{2}}(\partial\Omega_H) \;/\; \psi = 0 \text{ on } \Sigma_0 \qquad D_\varepsilon^{ss}\psi = \psi \implies \psi = 0$$