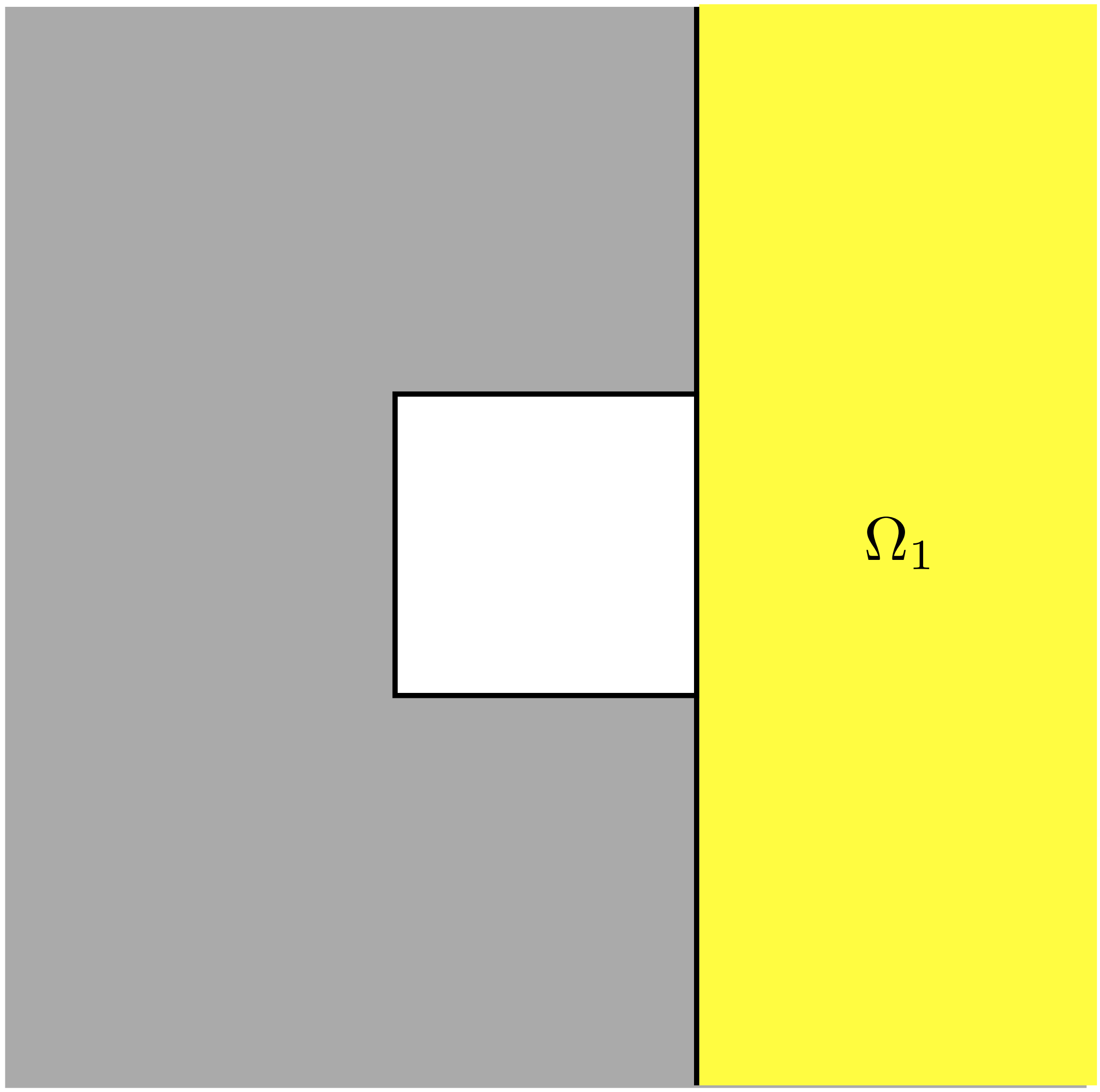
The uniqueness result

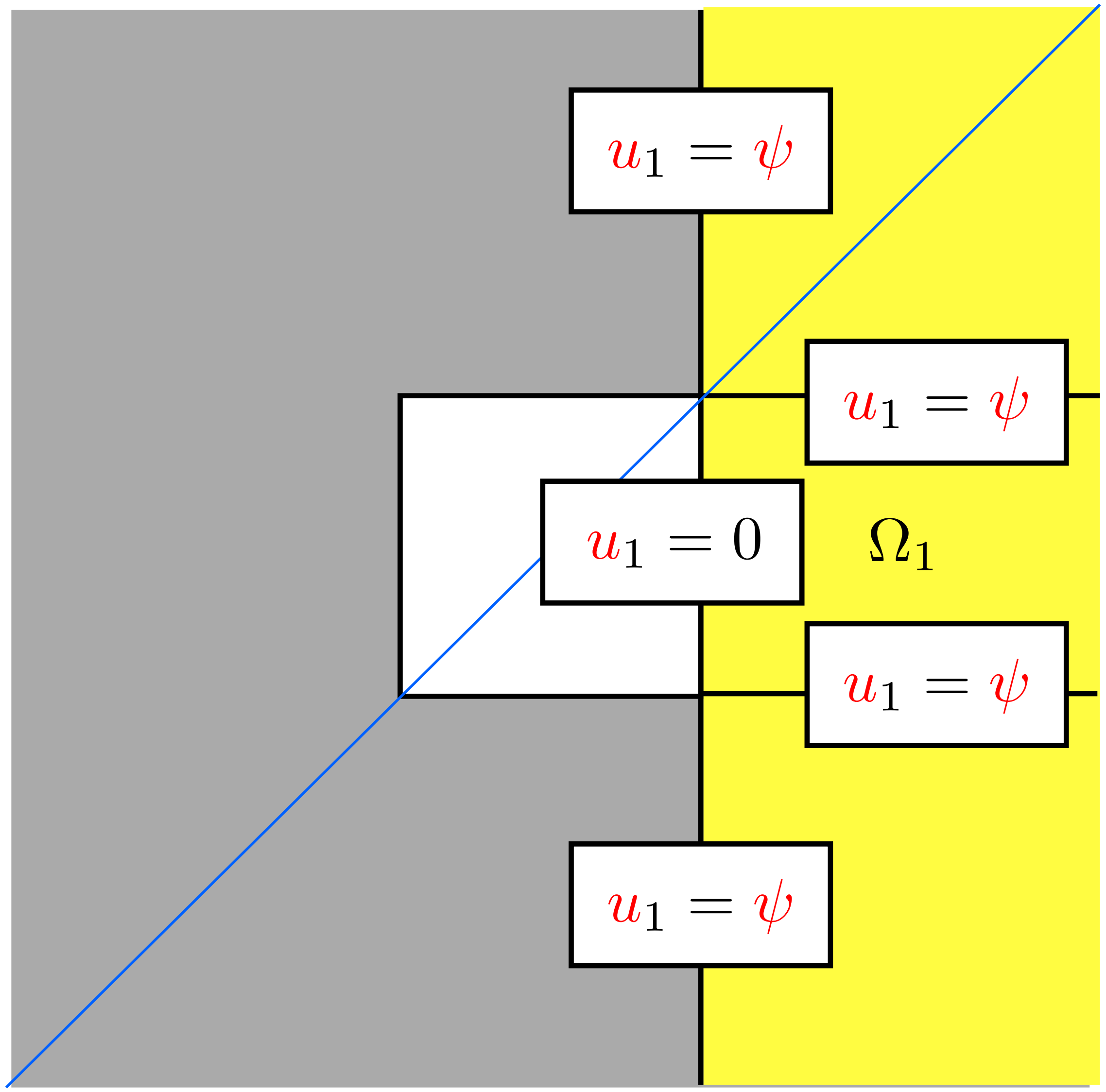


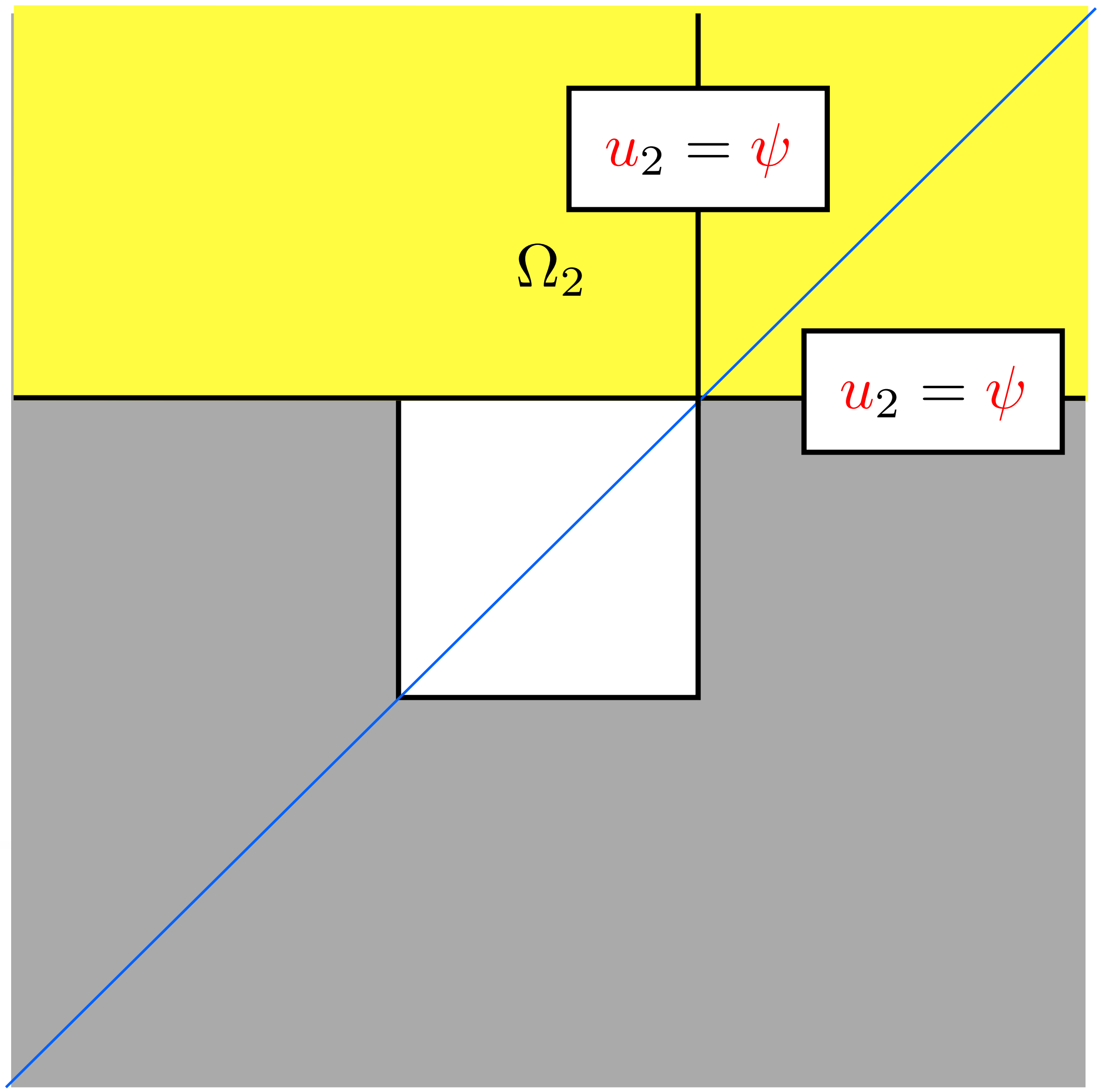
$\Omega_1 \equiv \Omega_H$

Let $\psi \in H^{\frac{1}{2}}(\partial\Omega_H)$ such that $\psi|_{\Sigma_0} = 0$

$$\boxed{D_\varepsilon^{ss}\psi = \psi \overset{?}{\Longrightarrow} \psi = 0}$$

The uniqueness result



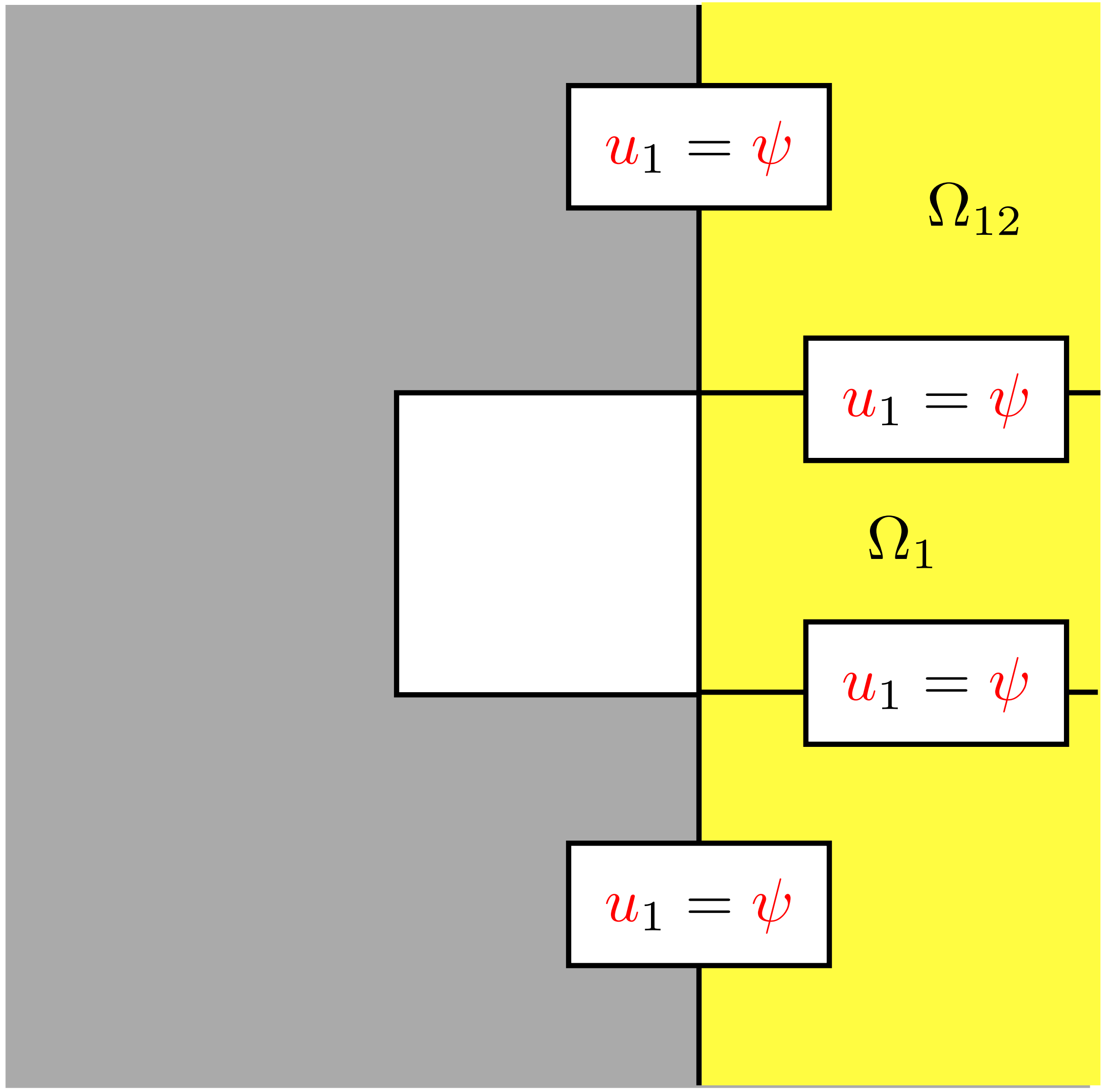


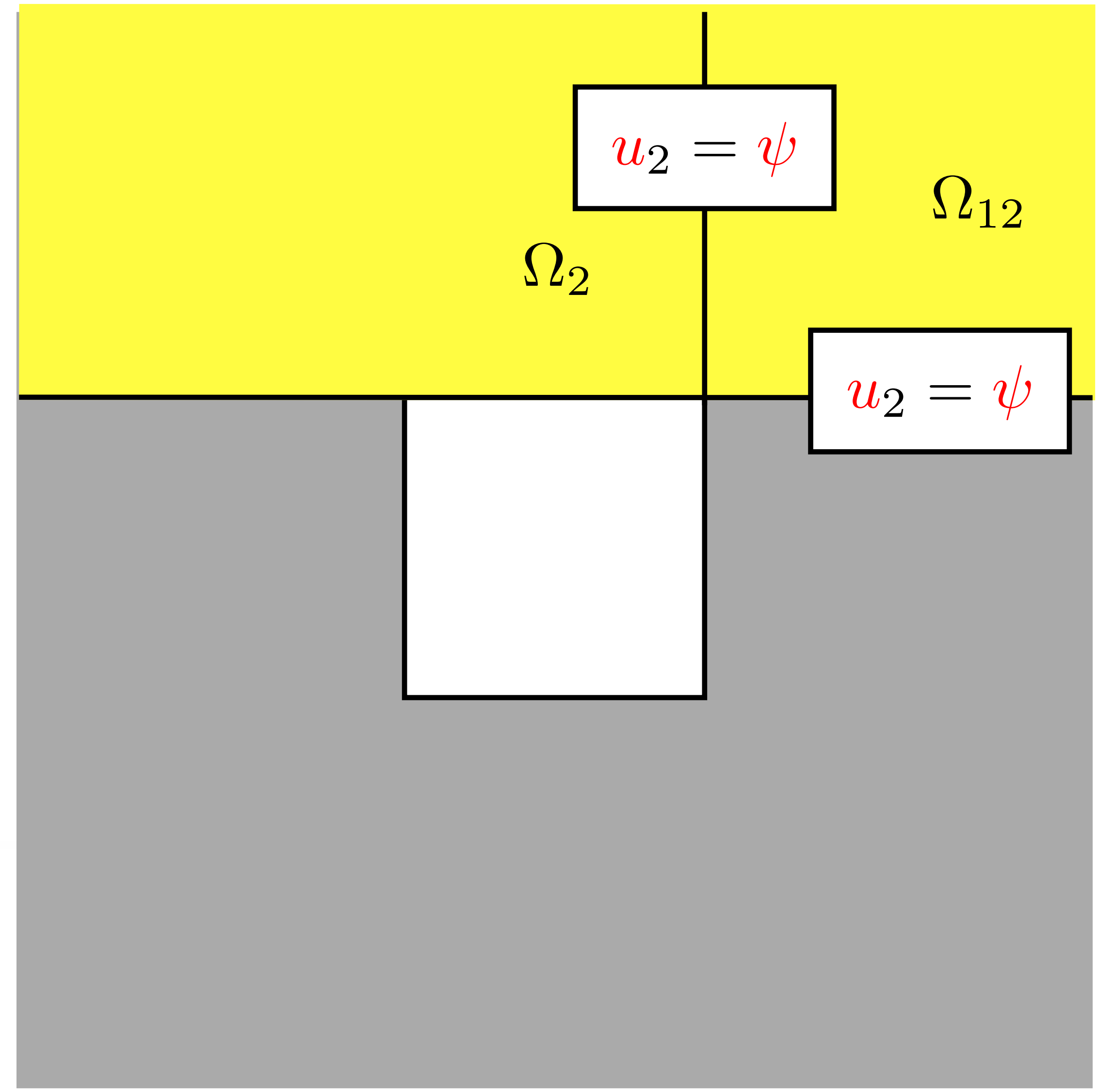
Let $\psi \in H^{\frac{1}{2}}(\partial\Omega_H)$ such that $\psi|_{\Sigma_0} = 0$

$$\boxed{u_1 := u_\varepsilon^H(\psi) \quad \text{in } \Omega_1}$$

$$\boxed{u_2 := u_1 \circ S_1 \quad \text{in } \Omega_2}$$

The uniqueness result





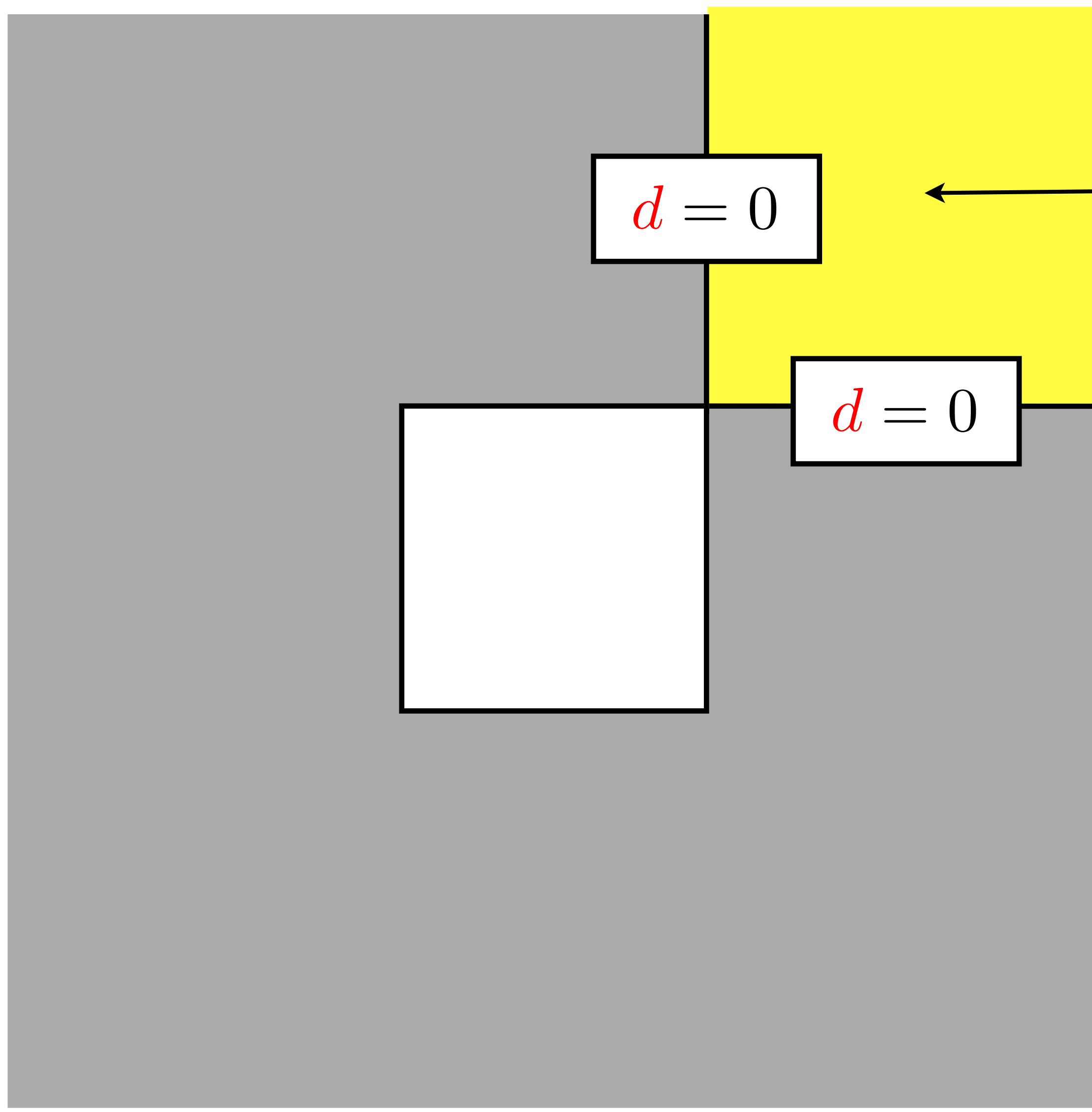
$$-\Delta u_1 - \rho_{per}(\omega^2 + \imath\varepsilon\omega)\, u_1 = 0 \qquad -\Delta u_2 - \rho_{per}(\omega^2 + \imath\varepsilon\omega)\, u_2 = 0$$

$\Delta(u \circ S_1) = \Delta u \circ S_1$

ρ_{per} doubly symmetric

$$\boxed{d := u_1 - u_2 \quad \text{in } \Omega_{12}}$$

The uniqueness result



$$-\Delta d - \rho_{per}(\omega^2 + \imath\varepsilon\omega)\, d = 0$$

$$\Big\Downarrow$$

uniqueness for the quarter plane problem

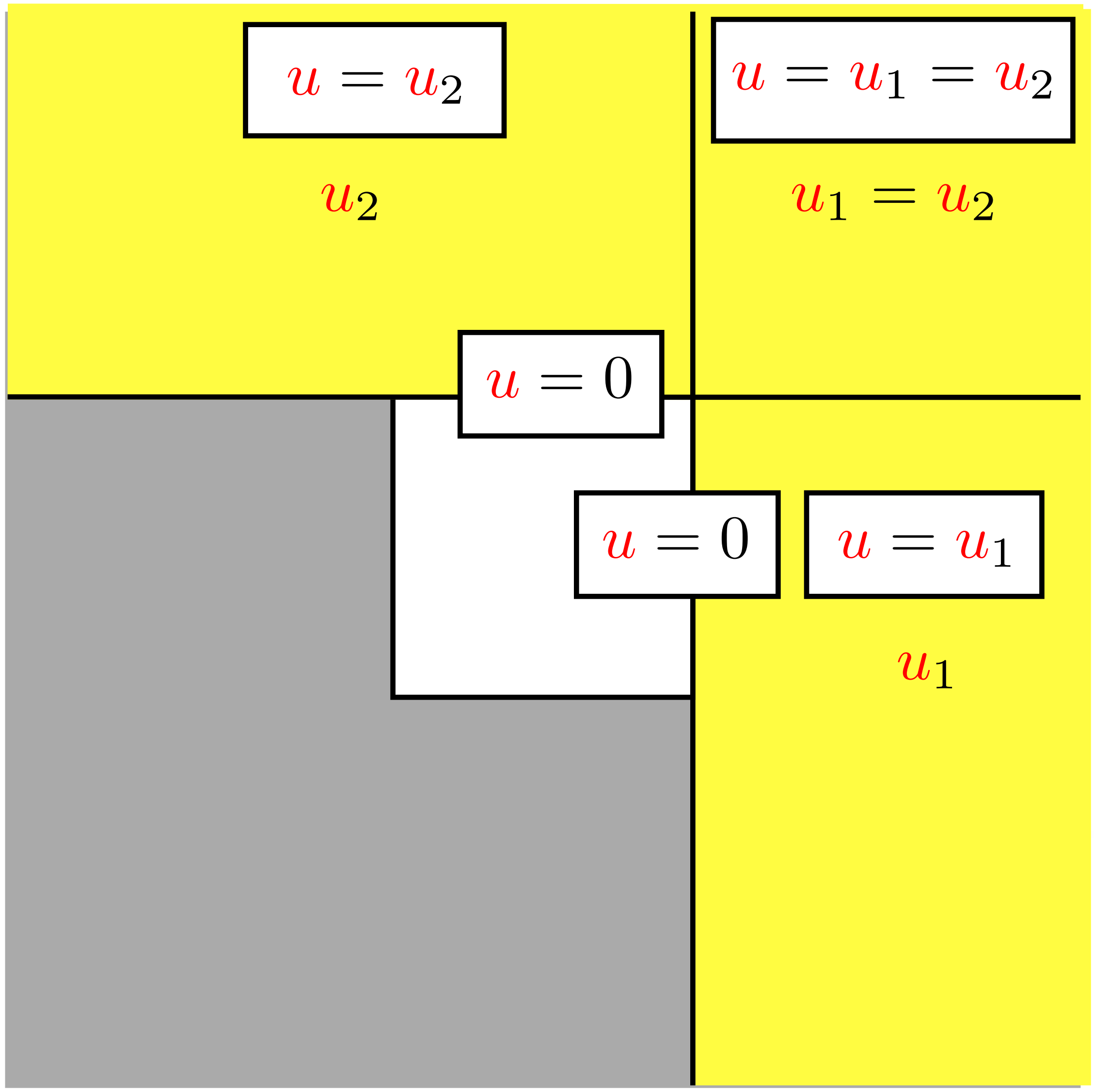
$$\Big\Downarrow$$

$$d = 0$$

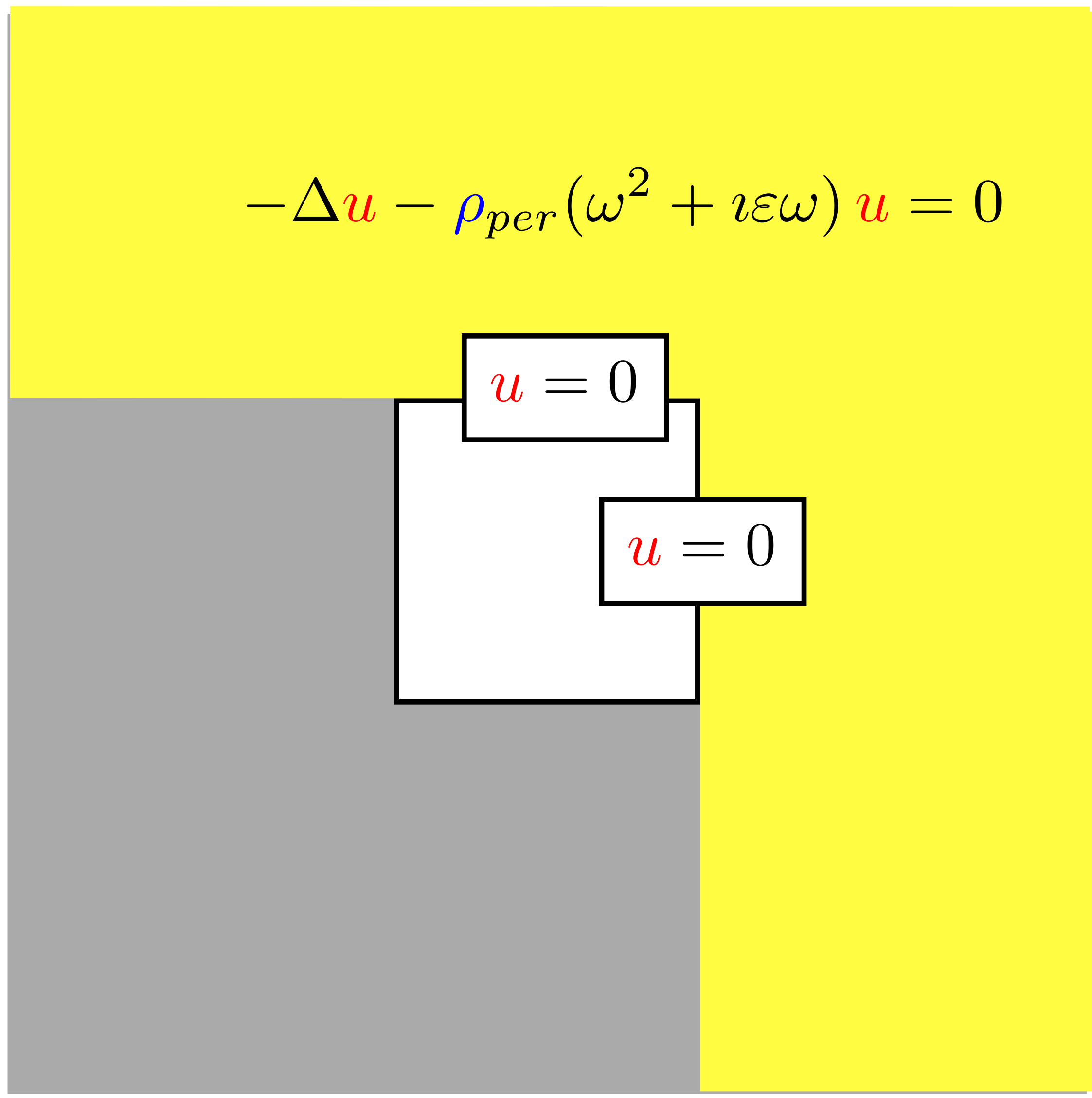
$$\Downarrow$$

$$u_1 = u_2$$

$$d := u_1 - u_2$$



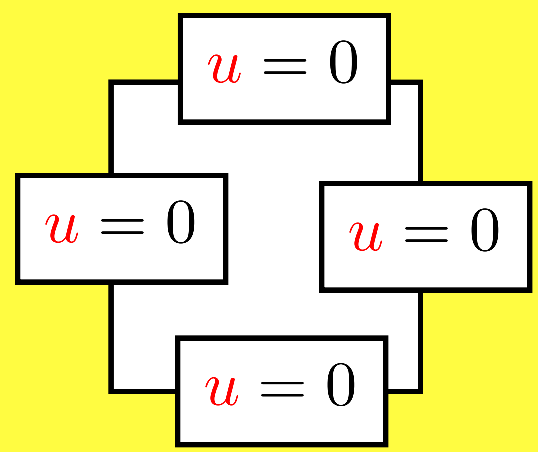
$$\boxed{u = u_1 = u_\varepsilon^H(\psi) \ \text{ in } \ \Omega_H}$$



$$\boxed{u = u_1 = u_\varepsilon^H(\psi) \ \text{ in } \ \Omega_H}$$

Repeating the same process with the second symmetry, we construct $u : \Omega_e \longrightarrow \mathbb{C}$ such that

$$-\Delta u - \rho_{per}(\omega^2 + \imath\varepsilon\omega)\, u = 0$$



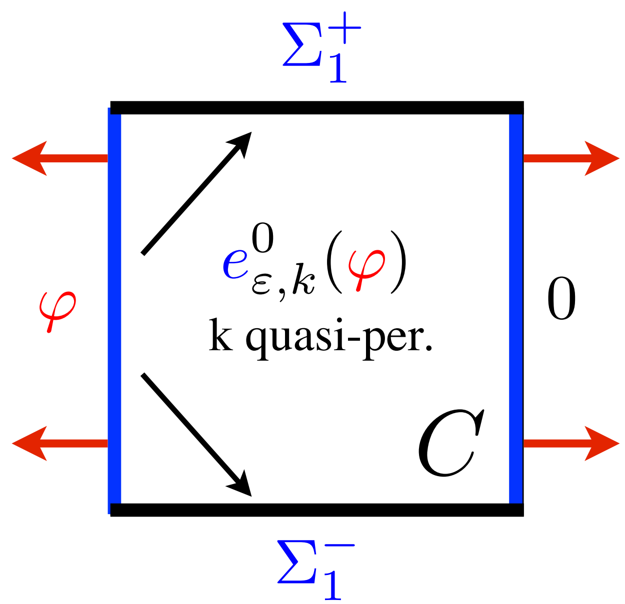
$\Longrightarrow$ $\boxed{u = 0}$

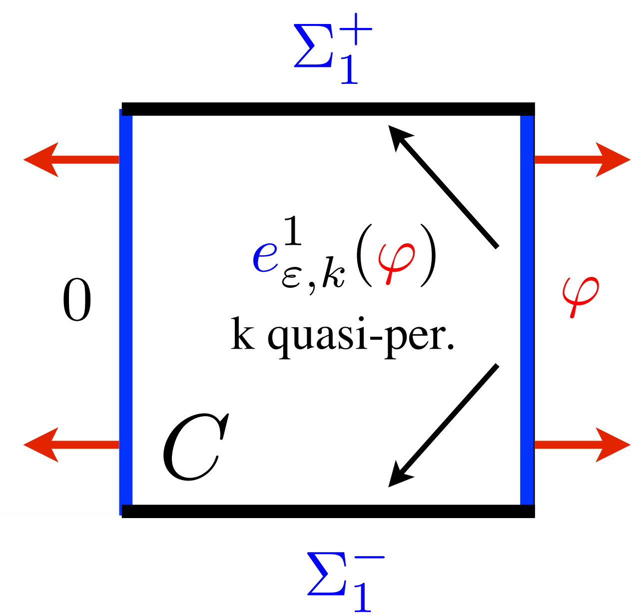
uniqueness for the exterior problem

$$\boxed{u = u_1 = u_\varepsilon^H(\psi) \text{ in } \Omega_H} \longrightarrow \psi = 0$$

Local Dirichlet to Dirichlet operators

$$\boxed{-\Delta e^{\ell}_{\varepsilon,k}(\varphi) - \rho_{\text{per}}(\mathbf{x})(\omega^2 + \imath\varepsilon\omega) e^{\ell}_{\varepsilon,k}(\varphi) = 0, \quad \text{in } \mathcal{C}}$$





$$D^{0,+}_{\varepsilon,k}(\varphi) := e^{0}_{\varepsilon,k}(\varphi)|_{\Sigma_1^+} \qquad D^{1,-}_{\varepsilon,k}(\varphi) := e^{1}_{\varepsilon,k}(\varphi)|_{\Sigma_1^-}$$

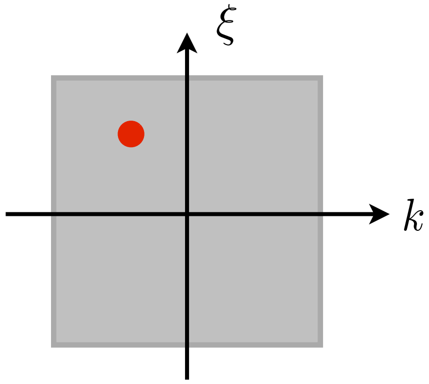
$$D^{0,-}_{\varepsilon,k}(\varphi) := e^{0}_{\varepsilon,k}(\varphi)|_{\Sigma_1^-} \qquad D^{1,+}_{\varepsilon,k}(\varphi) := e^{1}_{\varepsilon,k}(\varphi)|_{\Sigma_1^+}$$

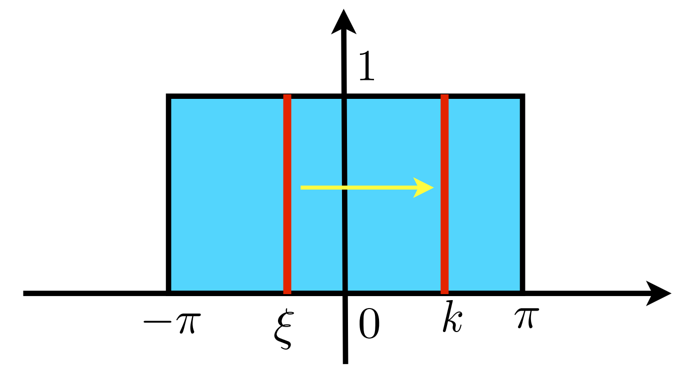
$$\boxed{D^{\ell,-}_{\varepsilon,k}(\varphi) :\equiv e^{ik}\, D^{\ell,-}_{\varepsilon,k}(\varphi). \quad \ell = 0, 1}$$

$$\begin{tikzcd}
H^{\frac{1}{2}}(\mathbb{R}) \arrow[r, "D_\varepsilon^{ss}"] \arrow[d, "\mathcal{F}"'] & H^{\frac{1}{2}}(\mathbb{R}) \arrow[d, "\mathcal{F}"] \\
H^{\frac{1}{2}}_{qp}(0,1) \arrow[r, "\widehat{D}_\varepsilon^{ss}"] & H^{\frac{1}{2}}_{qp}(0,1)
\end{tikzcd}$$

$\mathbb{R} \equiv \partial\Omega^H$

$\mathbb{R} \equiv \partial\Omega^+$





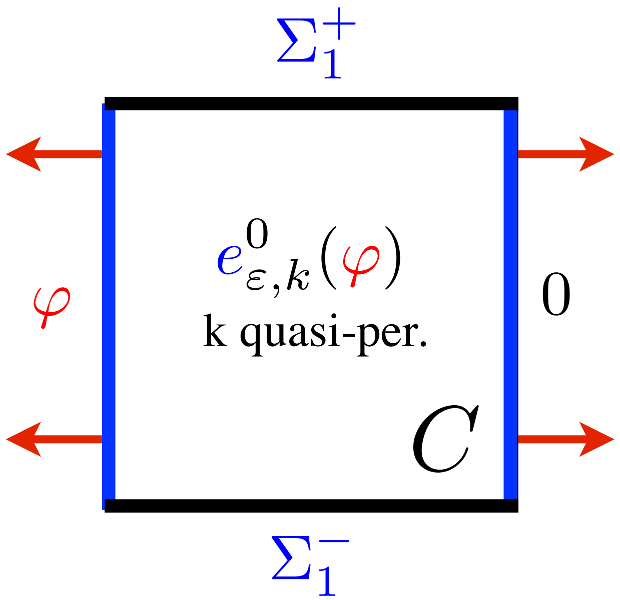
$$\boxed{D_\varepsilon^{ss} = \mathcal{F}^{-1} \circ \widehat{D}_\varepsilon^{ss} \circ \mathcal{F}}$$

$$\left(\widehat{D}_\varepsilon^{ss}\,\widehat{\varphi}\right)_k = \frac{1}{\sqrt{2\pi}} \int_{-\pi}^{\pi} \widehat{K}_\varepsilon^{ss}(\xi,k)\,\widehat{\varphi}_\xi\,d\xi \qquad \widehat{K}_\varepsilon^{ss}(\xi,k) \in \mathcal{L}\left(H_\xi^{\frac{1}{2}}(0.1), H_k^{\frac{1}{2}}(0.1)\right)$$

$$\widehat{K}_\varepsilon^{ss}(\xi,k) = I + \widehat{K}_{\varepsilon,+}^{ss}(\xi,k) + \widehat{K}_{\varepsilon,-}^{ss}(\xi,k)$$

$$\widehat{K}_{\varepsilon,\pm}^{ss}(\xi,k) = \quad e^{\mp ik}\left[D_{\varepsilon,\xi}^{1,\pm} + D_{\varepsilon,\xi}^{0,\pm}\,\mathcal{R}_{\varepsilon,\xi}\right]\left(I - e^{\mp ik}\,\mathcal{R}_{\varepsilon,\xi}\right)^{-1}$$

A computation expression for D_ε^{ss}

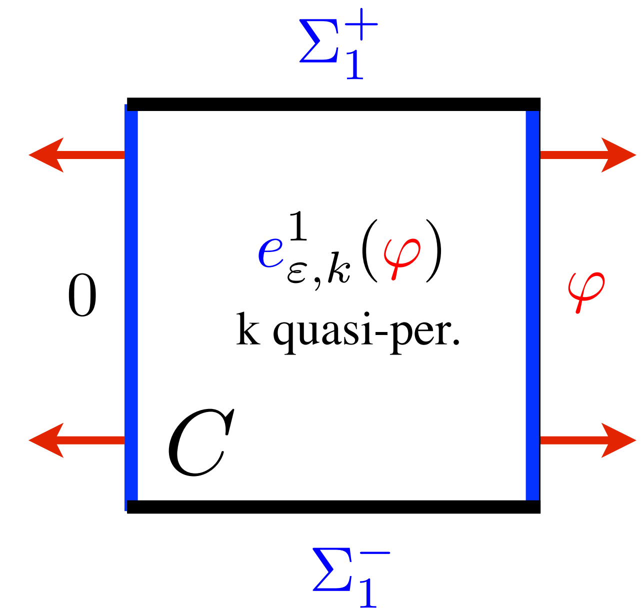


$$T^{00}_{\varepsilon,k}(\varphi) := \partial_n e^0_{\varepsilon,k}(\varphi)|_{\Sigma_0}$$

$$T^{01}_{\varepsilon,k}(\varphi) := \partial_n e^0_{\varepsilon,k}(\varphi)|_{\Sigma_1}$$

$$T^{10}_{\varepsilon,k}(\varphi) := \partial_n e^1_{\varepsilon,k}(\varphi)|_{\Sigma_0}$$

$$T^{11}_{\varepsilon,k}(\varphi) := \partial_n e^1_{\varepsilon,k}(\varphi)|_{\Sigma_1}$$



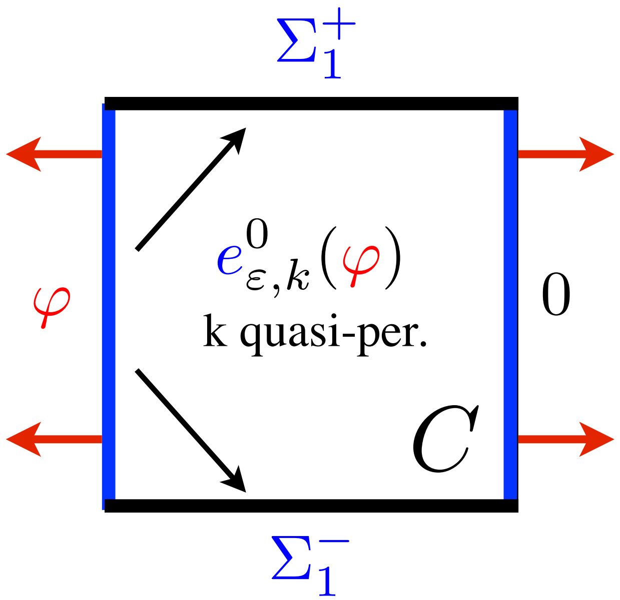
$$\boxed{T^{10}_{\varepsilon,k}\,\mathcal{R}^2 + (T^{00}_{\varepsilon,k} + T^{11}_{\varepsilon,k})\,\mathcal{R} + T^{01}_{\varepsilon,k} = 0}$$

$$\boxed{D_\varepsilon^{ss} = \mathcal{F}^{-1} \circ \widehat{D}_\varepsilon^{ss} \circ \mathcal{F}}$$

$$\left(\widehat{D}_\varepsilon^{ss}\,\widehat{\varphi}\right)_k = \frac{1}{\sqrt{2\pi}} \int_{-\pi}^{\pi} \widehat{K}_\varepsilon^{ss}(\xi, k)\,\widehat{\varphi}_\xi\, d\xi \qquad \widehat{K}_\varepsilon^{ss}(\xi, k) \in \mathcal{L}\big(H_\xi^{\frac{1}{2}}(0.1), H_k^{\frac{1}{2}}(0.1)\big)$$

$$\widehat{K}_\varepsilon^{ss}(\xi, k) = I + \widehat{K}_{\varepsilon,+}^{ss}(\xi, k) + \widehat{K}_{\varepsilon,-}^{ss}(\xi, k)$$

$$\widehat{K}_{\varepsilon,\pm}^{ss}(\xi, k) = e^{\mp ik}\left[D^{1,\pm}_{\varepsilon,\xi} + D^{0,\pm}_{\varepsilon,\xi}\,\mathcal{R}_{\varepsilon,\xi}\right]\left(I - e^{\mp ik}\,\mathcal{R}_{\varepsilon,\xi}\right)^{-1}$$

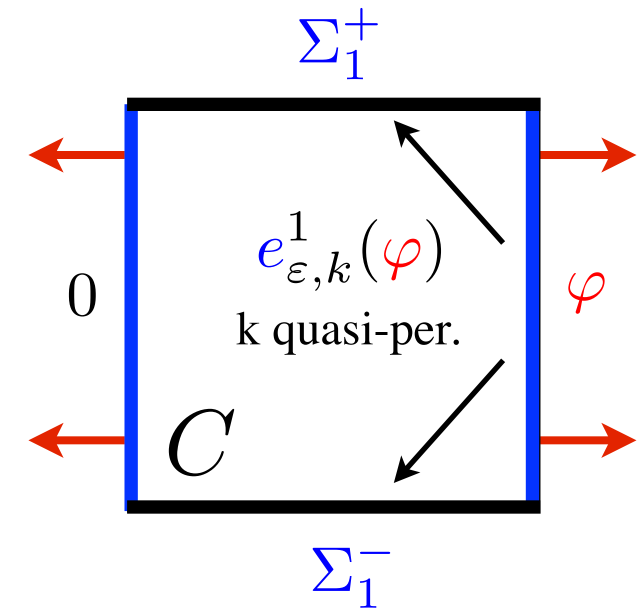


$$D^{0,+}_{\varepsilon,k}(\varphi) := e^0_{\varepsilon,k}(\varphi)|_{\Sigma_1^+}$$

$$D^{0,-}_{\varepsilon,k}(\varphi) := e^0_{\varepsilon,k}(\varphi)|_{\Sigma_1^-}$$

$$D^{1,-}_{\varepsilon,k}(\varphi) := e^1_{\varepsilon,k}(\varphi)|_{\Sigma_1^-}$$

$$D^{1,+}_{\varepsilon,k}(\varphi) := e^1_{\varepsilon,k}(\varphi)|_{\Sigma_1^+}$$



$$\boxed{D_\varepsilon^{ss} = \mathcal{F}^{-1} \circ \widehat{D}_\varepsilon^{ss} \circ \mathcal{F}}$$

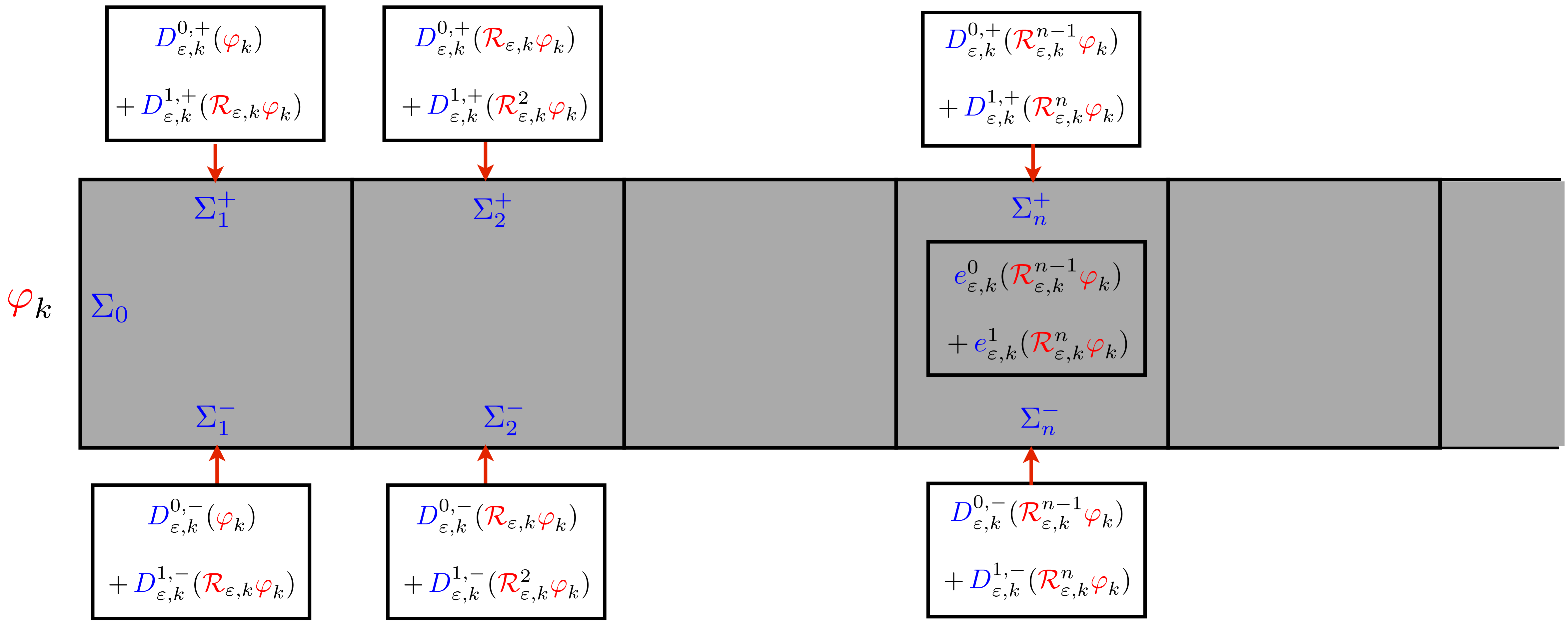
$$\left(\widehat{D}_\varepsilon^{ss}\,\widehat{\varphi}\right)_k = \frac{1}{\sqrt{2\pi}} \int_{-\pi}^{\pi} \widehat{K}_\varepsilon^{ss}(\xi,k)\,\widehat{\varphi}_\xi\,d\xi \qquad \widehat{K}_\varepsilon^{ss}(\xi,k) \in \mathcal{L}\big(H_\xi^{\frac{1}{2}}(0.1), H_k^{\frac{1}{2}}(0.1)\big)$$

$$\widehat{K}_\varepsilon^{ss}(\xi,k) = I + \widehat{K}_{\varepsilon,+}^{ss}(\xi,k) + \widehat{K}_{\varepsilon,-}^{ss}(\xi,k)$$

$$\widehat{K}_{\varepsilon,\pm}^{ss}(\xi,k) = \quad e^{\mp ik}\left[D^{1,\pm}_{\varepsilon,\xi} + D^{0,\pm}_{\varepsilon,\xi}\,\mathcal{R}_{\varepsilon,\xi}\right]\left(I - e^{\mp ik}\,\mathcal{R}_{\varepsilon,\xi}\right)^{-1}$$

A computable expression for D_ε^{ss}

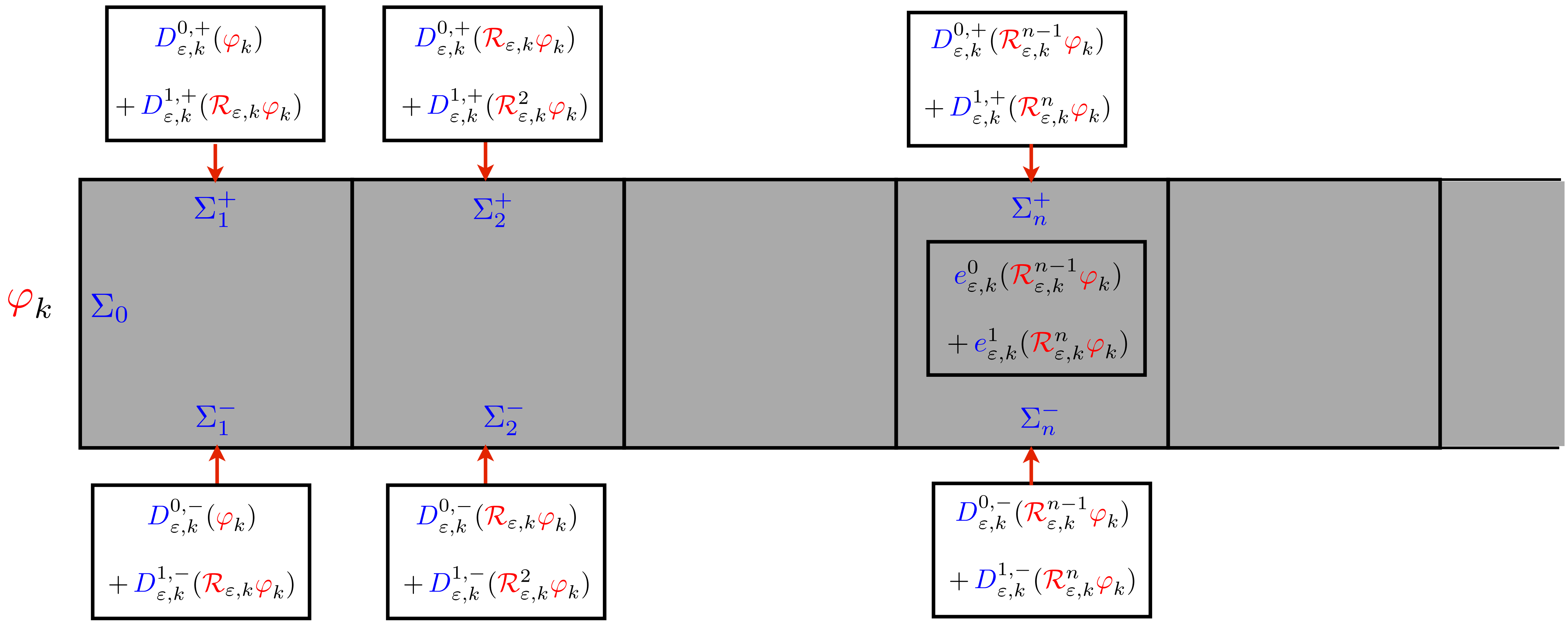
$$\boxed{\widehat{u}^H_{\varepsilon,k} = \mathcal{F}_y u^H_\varepsilon(\cdot, k)}$$



$$\boxed{\text{For } 0 \leq y \leq 1, \quad u^H_\varepsilon = \frac{1}{\sqrt{2\pi}} \int_{-\pi}^{\pi} \widehat{u}^H_{\varepsilon,\xi}\, d\xi}$$

A computable expression for D_ε^{ss}

$$\text{For } 0 \leq y \leq 1, \quad u_\varepsilon^H = \frac{1}{\sqrt{2\pi}} \int_{-\pi}^{\pi} \widehat{u}_{\varepsilon,\xi}^H \, d\xi$$



$$u_\varepsilon^H|_{\Sigma_0} = \varphi$$

$$u_\varepsilon^H|_{\Sigma_n^\pm} = \frac{1}{\sqrt{2\pi}} \int_{-\pi}^{\pi} \left(D_{\varepsilon,\xi}^{0,\pm}\, \mathcal{R}_{\varepsilon,\xi}^{n-1} + D_{\varepsilon,\xi}^{1,\pm}\, \mathcal{R}_{\varepsilon,\xi}^{n} \right) \varphi_\xi \, d\xi$$

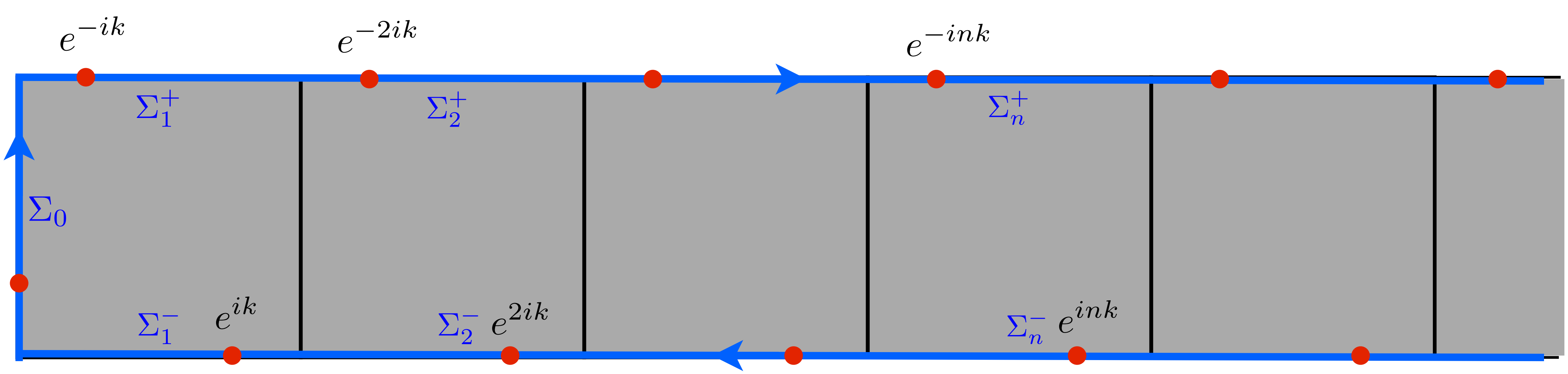
A computable expression for D_ε^{ss}

$$\boxed{u_\varepsilon^H|_{\Sigma_0} = \varphi}$$

$$\boxed{u_\varepsilon^H|_{\Sigma_n^\pm} = \frac{1}{\sqrt{2\pi}} \int_{-\pi}^{\pi} \left(D_{\varepsilon,\xi}^{0,\pm}\, \mathcal{R}_{\varepsilon,\xi}^{n-1} + D_{\varepsilon,\xi}^{1,\pm}\, \mathcal{R}_{\varepsilon,\xi}^{n} \right) \varphi_\xi \, d\xi}$$

$$\boxed{D_\varepsilon^{ss}\varphi = u_\varepsilon^H|_{\partial\Omega^+}}$$

$$\left(\widehat{D_\varepsilon^{ss}\varphi}\right)_k = \varphi + \sum_{n=1}^{+\infty} \left(u_\varepsilon^H|_{\Sigma_n^+} \right) e^{-ink} + \sum_{n=1}^{+\infty} \left(u_\varepsilon^H|_{\Sigma_n^-} \right) e^{ink}$$



$$\sum_{n=1}^{+\infty} e^{\mp ink}\, D_{\varepsilon,\xi}^{1,\pm}\, \mathcal{R}_{\varepsilon,\xi}^{n} = e^{\mp ik}\, D_{\varepsilon,\xi}^{1,\pm}\, \mathcal{R}_{\varepsilon,\xi} \left(I - e^{\mp ik}\, \mathcal{R}_{\varepsilon,\xi} \right)^{-1}$$

+ Fubini

$$\sum_{n=1}^{+\infty} e^{\mp ink}\, D_{\varepsilon,\xi}^{0,\pm}\, \mathcal{R}_{\varepsilon,\xi}^{n-1} = D_{\varepsilon,\xi}^{0,\pm} \left(I - e^{\mp ik}\, \mathcal{R}_{\varepsilon,\xi} \right)^{-1}$$

$$\boxed{\rho\left(\mathcal{R}_{\varepsilon,\xi}\right) \leq \tau_\varepsilon < 1}$$