Numerical methods for time harmonic locally perturbed periodic media

Part 4

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Numerical methods for time harmonic scalar wave equation in locally perturbed periodic media - Part 4

Sonia Fliss

POEMS (UMR 7231 CNRS-INRIA-ENSTA)

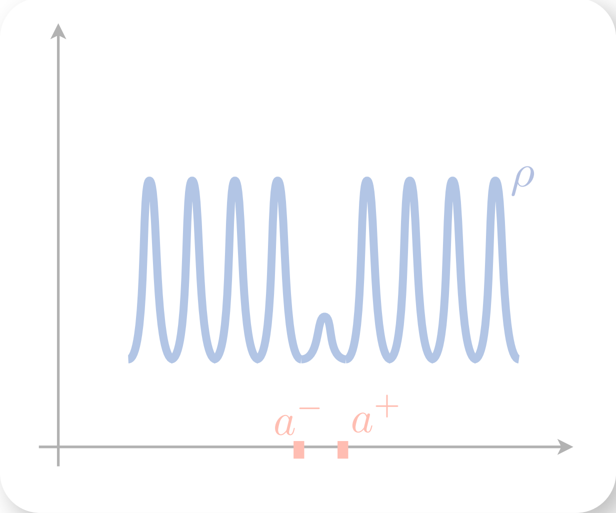
Mainly based on joint works with Julien Coatleven, Patrick Joly and Jing-Rebecca Li

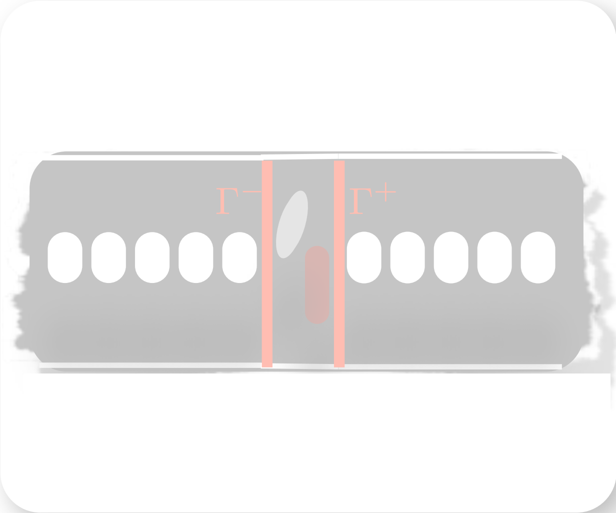
1D Case

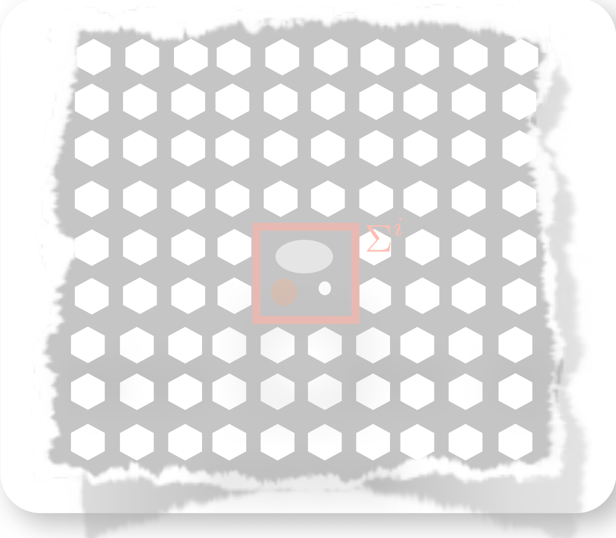
Waveguide case

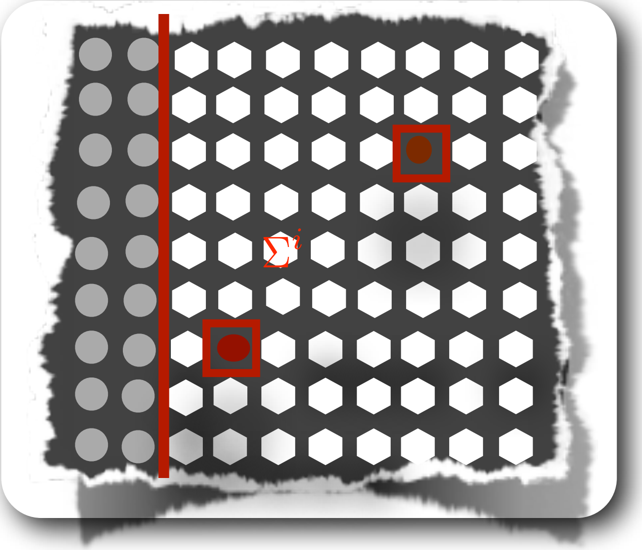
2D case
Simple scattering

2D case
Multiple scattering







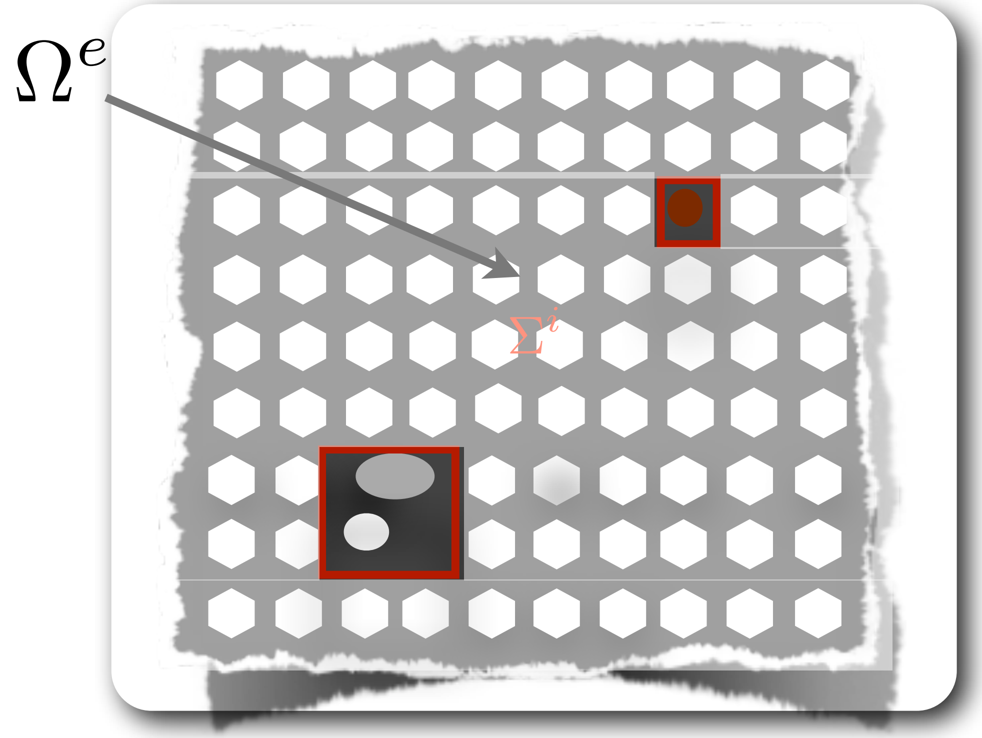


The 2D plane problem - Multiple scattering

Time harmonic Scalar wave Problem with absorption

$$(\mathcal{P}_\varepsilon^i) \quad \left| \begin{array}{l} -\triangle u_\varepsilon^i - \rho(\mathbf{x})(\omega^2 + \imath\varepsilon\omega)u_\varepsilon^i = f(\mathbf{x}), \quad \text{in } \Omega^i \\ \dfrac{\partial u_\varepsilon^i}{\partial \mathbf{n}} + \Lambda_\varepsilon u_\varepsilon^i = 0 \quad \text{on } \Sigma^i = \partial\Omega^i \end{array} \right.$$

$$\Omega^i = \Omega_1^i \cup \Omega_2^i \qquad \Sigma^i = \Sigma_1^i \cup \Sigma_2^i$$



2D case - simple scattering

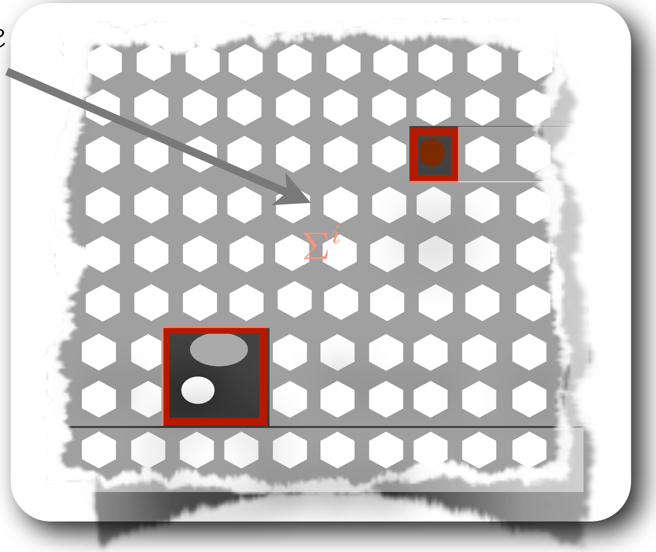
$\forall \varphi \in H^{1/2}(\Sigma^i)$, $u_\epsilon^e(\varphi)$ is the unique H^1solution of the exterior problem

$$(\mathcal{P}_\varepsilon^e) \quad \left| \begin{array}{l} -\triangle u_\varepsilon^e - \rho_{per}(\mathbf{x})(\omega^2 + \imath\varepsilon\omega)u_\varepsilon^e = 0, \quad \text{in } \Omega^e \\ u_\varepsilon^e = \varphi, \quad \text{on } \Sigma^i \end{array} \right.$$

$$\Lambda_\varepsilon \varphi = \frac{\partial\, u_\varepsilon^e(\varphi)}{\partial\, \mathbf{n}} \Big|_{\Sigma^i}$$

The 2D plane problem - Multiple scattering

Ω^e



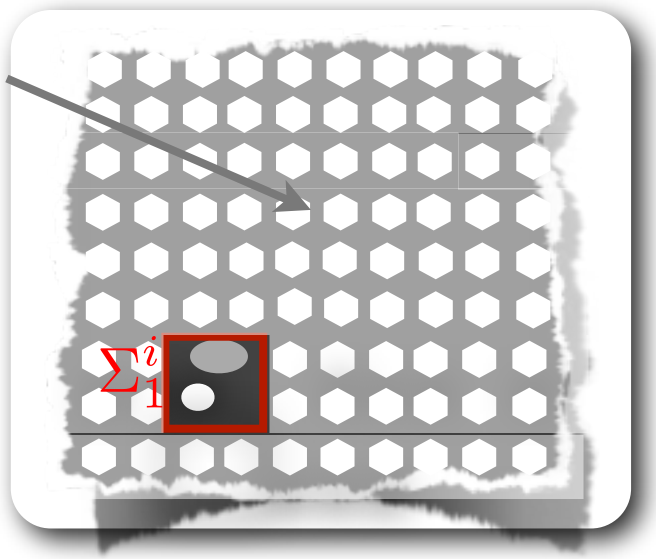
2D case - simple scattering

$\forall \varphi \in H^{1/2}(\Sigma^i)$, $u^e_\epsilon(\varphi)$ is the unique H^1solution of the exterior problem

$$(\mathcal{P}^e_\varepsilon) \quad \left| \begin{array}{ll} -\triangle u^e_\varepsilon - \rho_{per}(\mathbf{x})(\omega^2 + \imath\varepsilon\omega)u^e_\varepsilon = 0, & \text{in } \Omega^e \\ u^e_\varepsilon = \varphi, & \text{on } \Sigma^i \end{array} \right.$$

$$\Lambda_\varepsilon \, \varphi = \left. \frac{\partial \, u^e_\varepsilon(\varphi)}{\partial \, \mathbf{n}} \right|_{\Sigma^i}$$

Ω^e_1



$\forall \varphi_1 \in H^{1/2}(\Sigma^i_1)$, $u^e_1(\varphi_1)$ is the unique H^1solution of the exterior problem posed in Ω^e_1

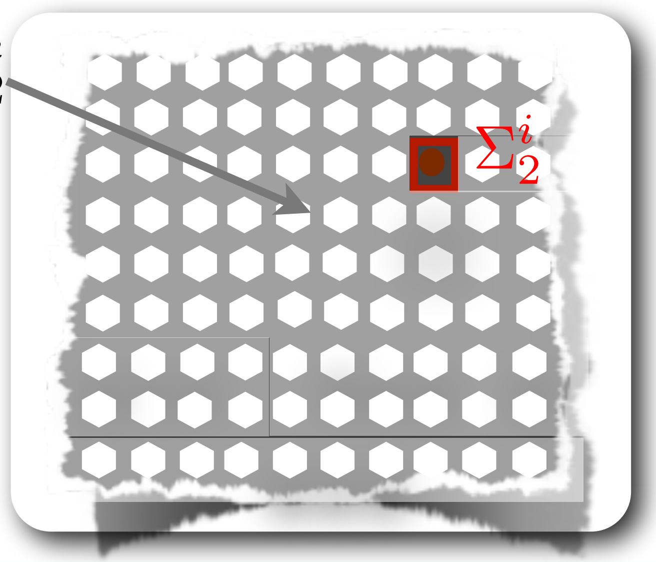
Λ_1 corresponding DtN operator

Theorem

If $\Sigma^i_1 \cap \Sigma^i_2 = \emptyset$, $\forall \varphi \in H^{1/2}(\Sigma^i), \exists!(\varphi_1, \varphi_2) \in H^{1/2}(\Sigma^i_1) \times H^{1/2}(\Sigma^i_2)$

$$u^e(\varphi) = u^e_1(\varphi_1)\Big|_{\Omega^e} + u^e_2(\varphi_2)\Big|_{\Omega^e}$$

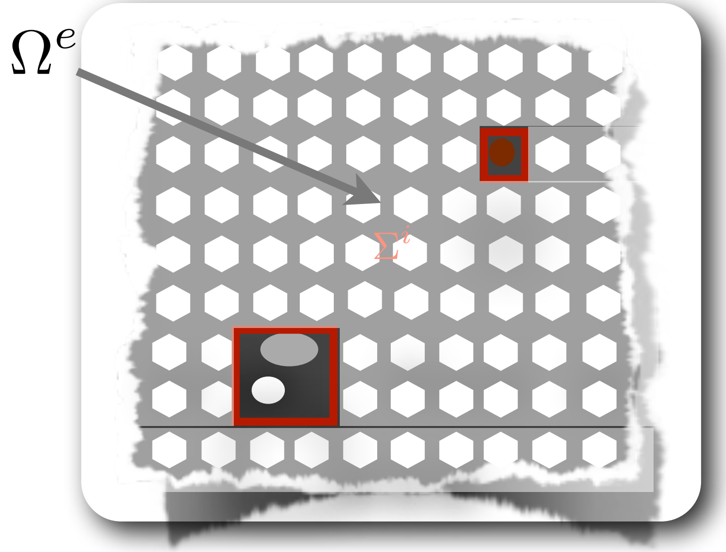
Ω^e_2



$\forall \varphi_2 \in H^{1/2}(\Sigma^i_2)$, $u^e_2(\varphi_2)$ is the unique H^1solution of the exterior problem posed in Ω^e_2

Λ_2 corresponding DtN operator

The 2D plane problem - Multiple scattering



2D case - simple scattering

$\forall \varphi \in H^{1/2}(\Sigma^i)$, $u^e_\varepsilon(\varphi)$ is the unique H^1 solution of the exterior problem

$$(\mathcal{P}^e_\varepsilon) \quad \left| \begin{array}{ll} -\triangle u^e_\varepsilon - \rho_{per}(\mathbf{x})(\omega^2 + \imath\varepsilon\omega)u^e_\varepsilon = 0, & \text{in } \Omega^e \\ u^e_\varepsilon = \varphi, & \text{on } \Sigma^i \end{array} \right.$$

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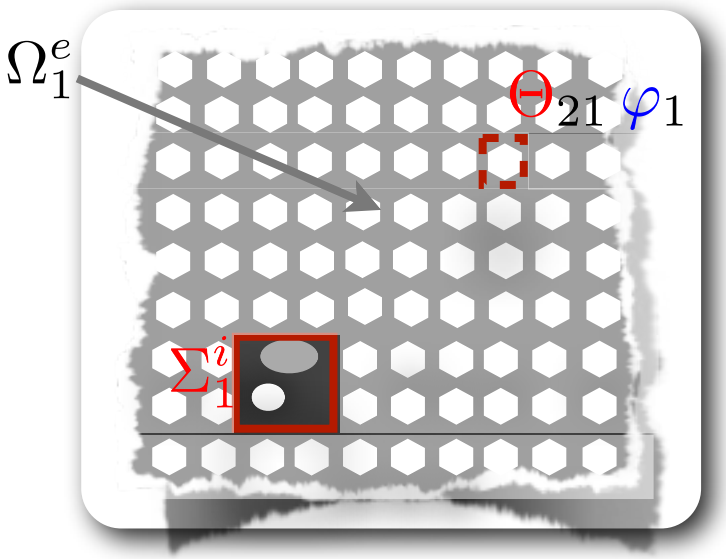
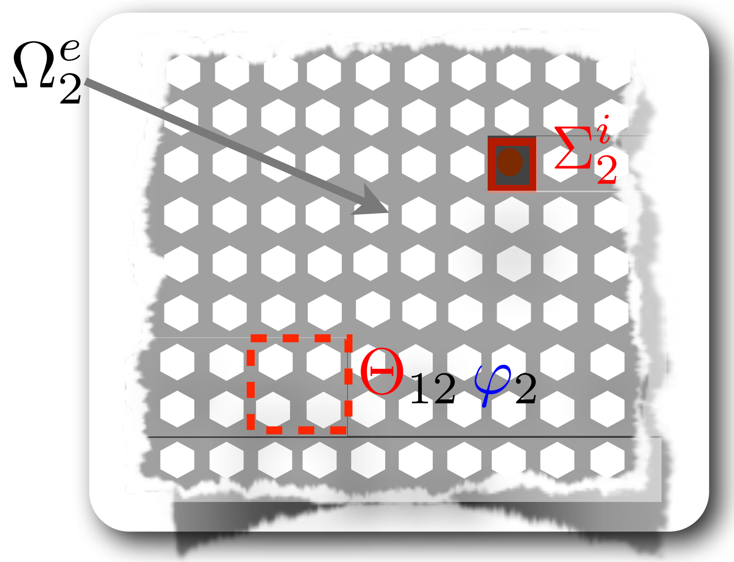


Theorem

If $\Sigma^i_1 \cap \Sigma^i_2 = \emptyset$, $\forall \varphi \in H^{1/2}(\Sigma^i), \exists!(\varphi_1, \varphi_2) \in H^{1/2}(\Sigma^i_1) \times H^{1/2}(\Sigma^i_2)$

$$u^e(\varphi) = u^e_1(\varphi_1)\Big|_{\Omega^e} + u^e_2(\varphi_2)\Big|_{\Omega^e}$$



$$\varphi\Big|_{\Sigma^i_1} = \varphi_1 + \underbrace{u^e_2(\varphi_2)\Big|_{\Sigma^i_1}}_{\Theta_{12}\,\varphi_2}$$

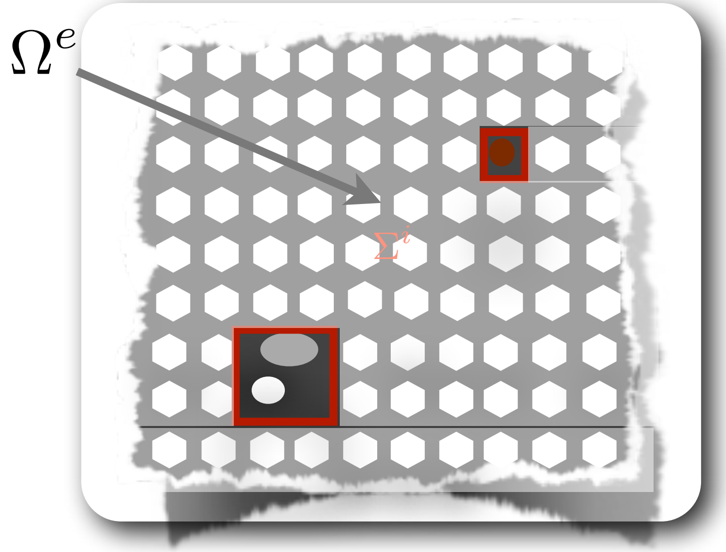
$$\varphi\Big|_{\Sigma^i_2} = \underbrace{u^e_1(\varphi_1)\Big|_{\Sigma^i_2}}_{\Theta_{21}\,\varphi_1} + \varphi_2$$

$\longrightarrow$

$$\left(\varphi\Big|_{\Sigma^i_1}, \varphi\Big|_{\Sigma^i_2}\right) = \Theta\,(\varphi_1, \varphi_2)$$

where $\Theta = \begin{bmatrix} \mathbb{I} & \Theta_{12} \\ \Theta_{21} & \mathbb{I} \end{bmatrix}$

The 2D plane problem - Multiple scattering



2D case - simple scattering

$\forall \varphi \in H^{1/2}(\Sigma^i)$, $u^e_\varepsilon(\varphi)$ is the unique H^1solution of the exterior problem

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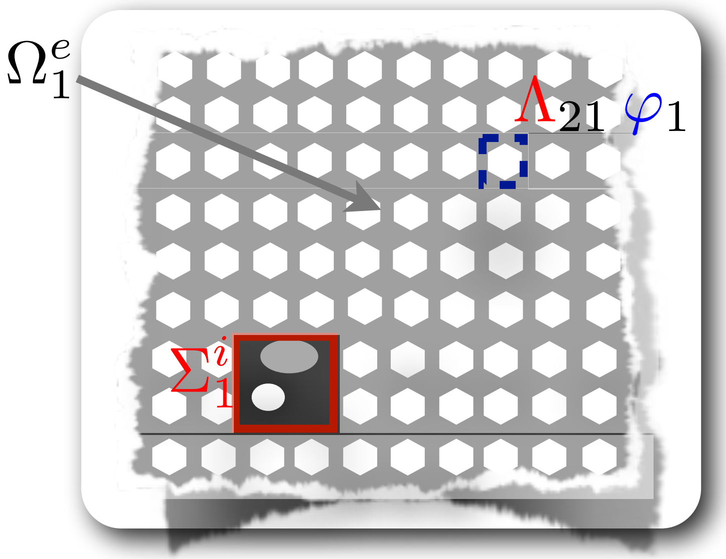

Theorem

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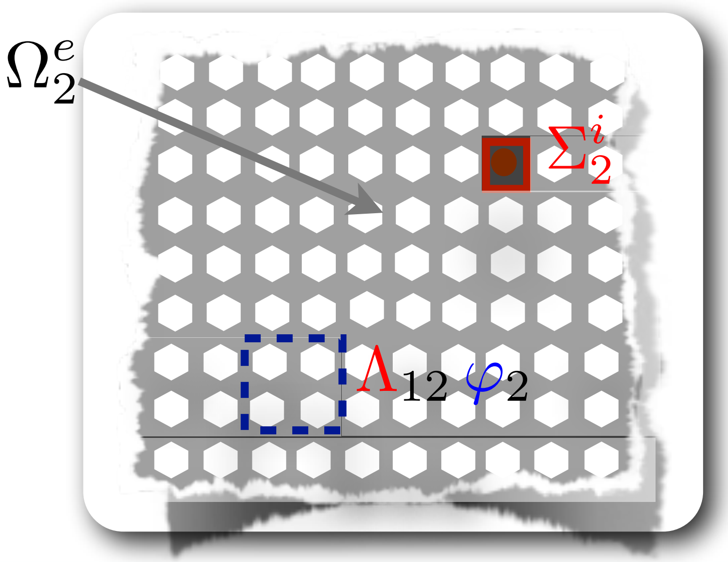


$$\Lambda_\varepsilon\varphi\Big|_{\Sigma^i_1} = \Lambda_1\varphi_1 + \underbrace{\frac{\partial}{\partial \mathbf{n}} u^e_2(\varphi_2)\Big|_{\Sigma^i_1}}_{\Lambda_{12}\,\varphi_2}$$

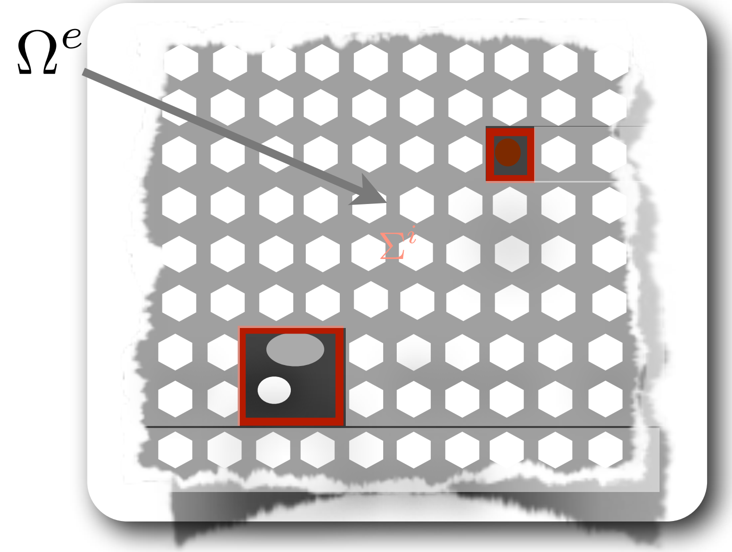
$$\Lambda_\varepsilon\varphi\Big|_{\Sigma^i_2} = \underbrace{\frac{\partial}{\partial \mathbf{n}} u^e_1(\varphi_1)\Big|_{\Sigma^i_2}}_{\Lambda_{21}\,\varphi_1} + \Lambda_2\varphi_2$$

$\rightarrow$

$$\left(\Lambda_\varepsilon\varphi\Big|_{\Sigma^i_1}, \Lambda_\varepsilon\varphi\Big|_{\Sigma^i_2}\right) = \Lambda\,(\varphi_1, \varphi_2)$$

where $\Lambda = \begin{bmatrix} \Lambda_1 & \Lambda_{12} \\ \Lambda_{21} & \Lambda_2 \end{bmatrix}$

The 2D plane problem - Multiple scattering

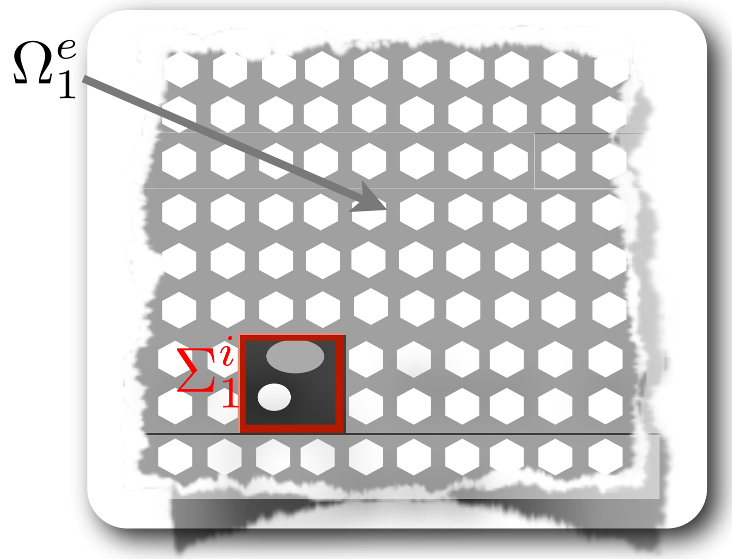


2D case - simple scattering

$\forall \varphi \in H^{1/2}(\Sigma^i)$, $u^e_\epsilon(\varphi)$ is the unique H^1solution of the exterior problem

$$(\mathcal{P}^e_\varepsilon) \quad \left| \begin{array}{ll} -\triangle u^e_\varepsilon - \rho_{per}(\mathbf{x})(\omega^2 + \imath\varepsilon\omega)u^e_\varepsilon = 0, & \text{in } \Omega^e \\ u^e_\varepsilon = \varphi, & \text{on } \Sigma^i \end{array} \right.$$

$$\Lambda_\varepsilon \, \varphi = \left. \frac{\partial \, u^e_\varepsilon(\varphi)}{\partial \, \mathbf{n}} \right|_{\Sigma^i}$$



Theorem

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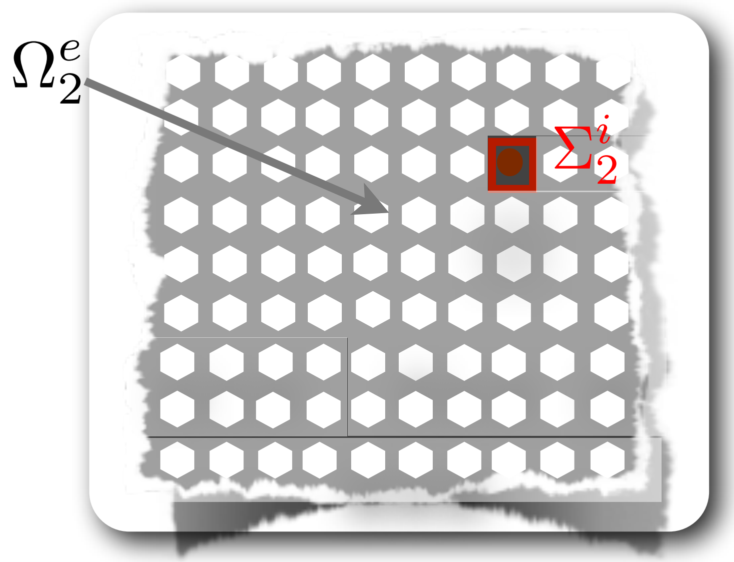
$$u^e(\varphi) = u^e_1(\varphi_1)\Big|_{\Omega^e} + u^e_2(\varphi_2)\Big|_{\Omega^e}$$



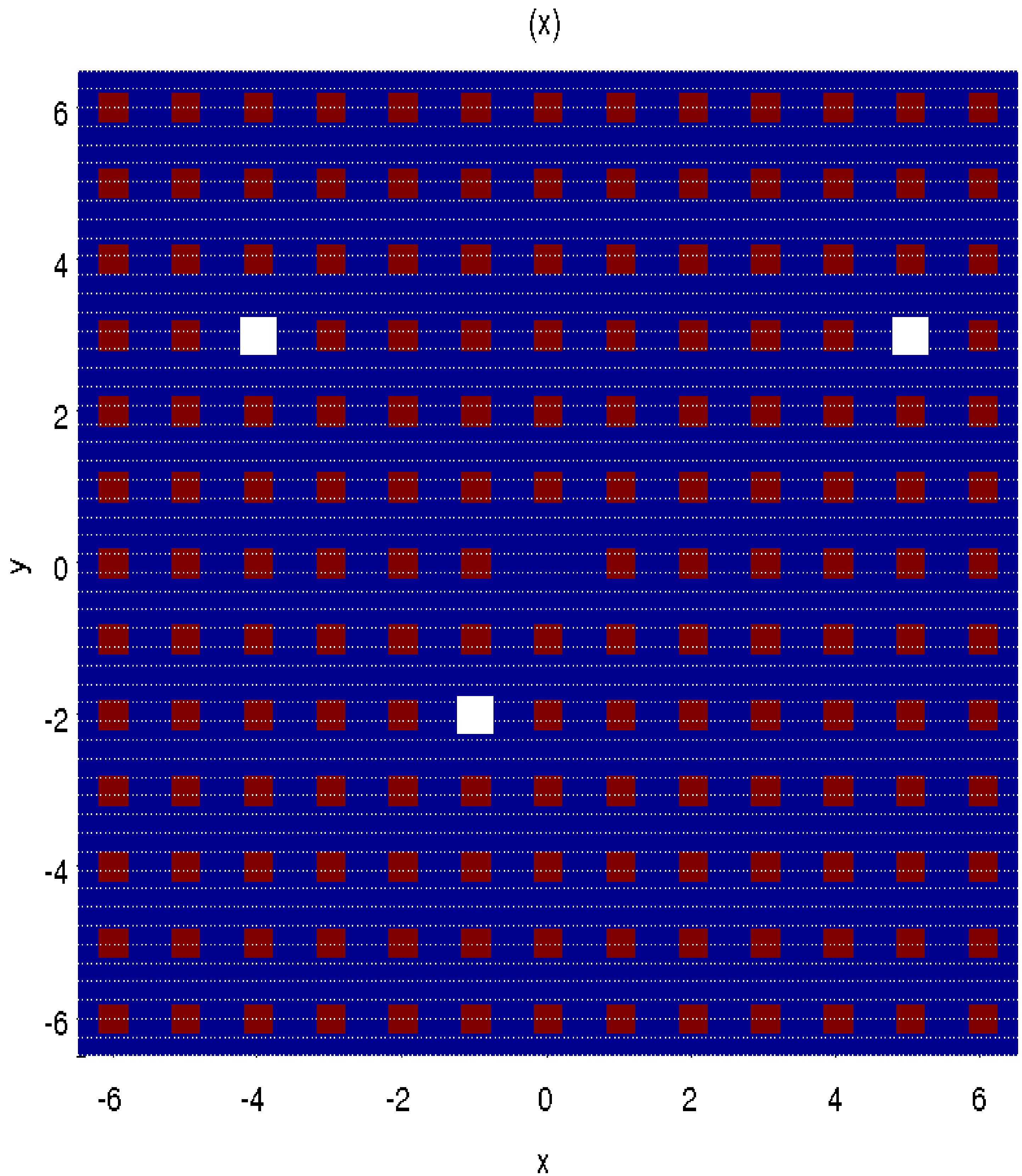
Theorem

$$\Lambda_\varepsilon = \Lambda \Theta^{-1}$$

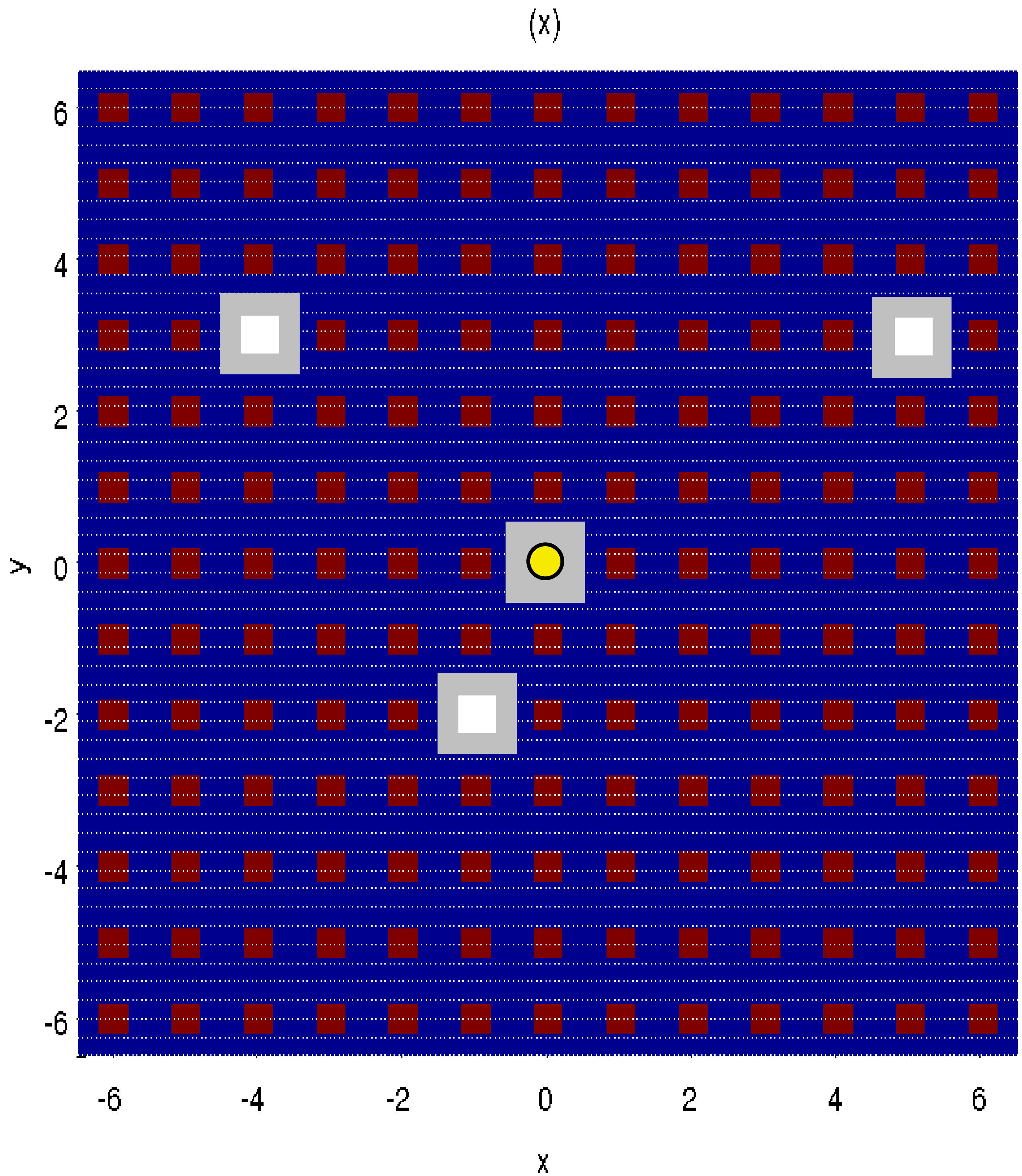
where $\Theta = \begin{bmatrix} \mathbb{I} & \Theta_{12} \\ \Theta_{21} & \mathbb{I} \end{bmatrix}$ and $\Lambda = \begin{bmatrix} \Lambda_1 & \Lambda_{12} \\ \Lambda_{21} & \Lambda_2 \end{bmatrix}$

For homogenous media : Balabane & Tirel (1997)**,** Grote & Kirsch (2004)**,** Ben Hassen et al. (2007)

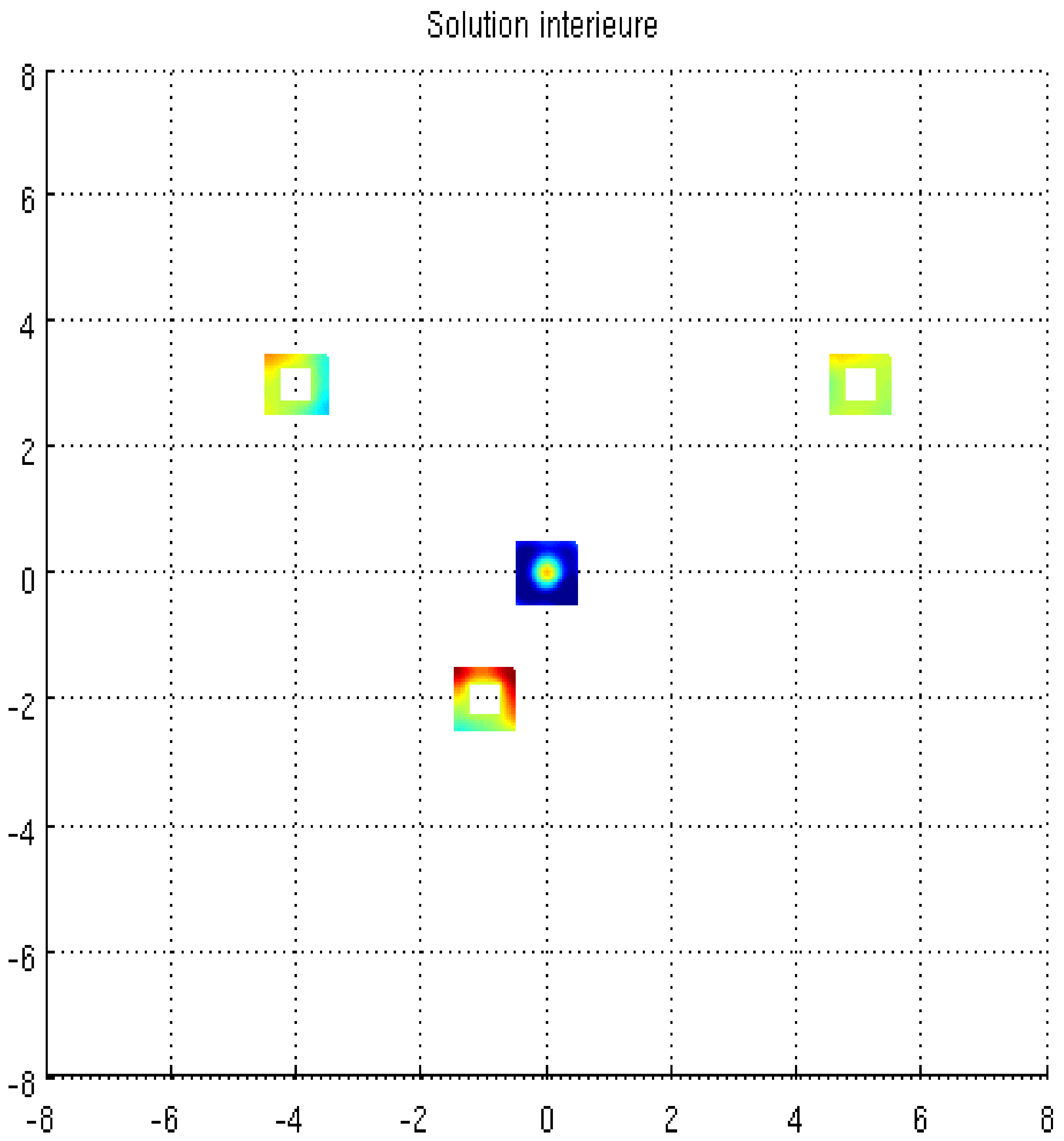
Numerical results



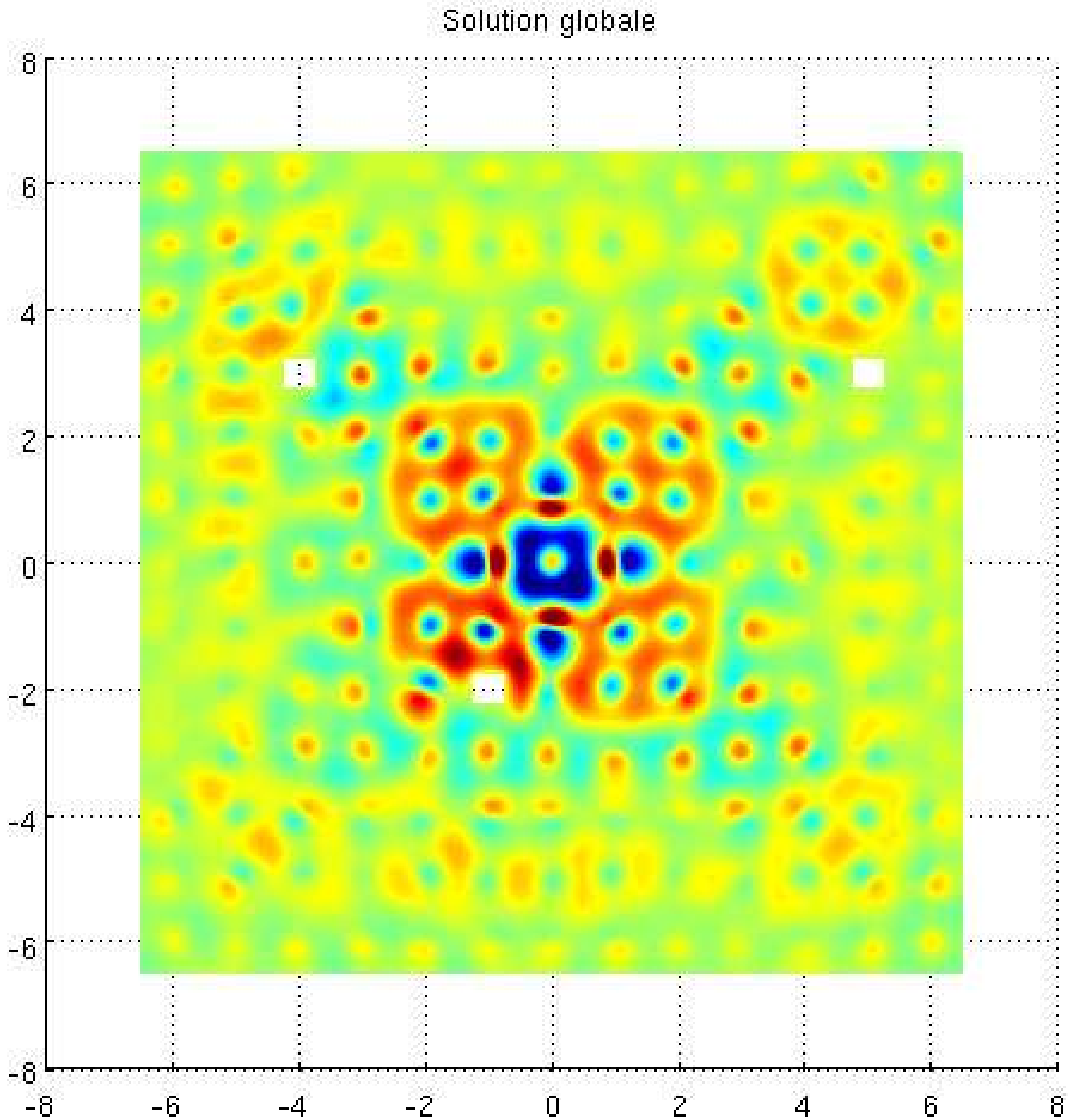
Numerical results



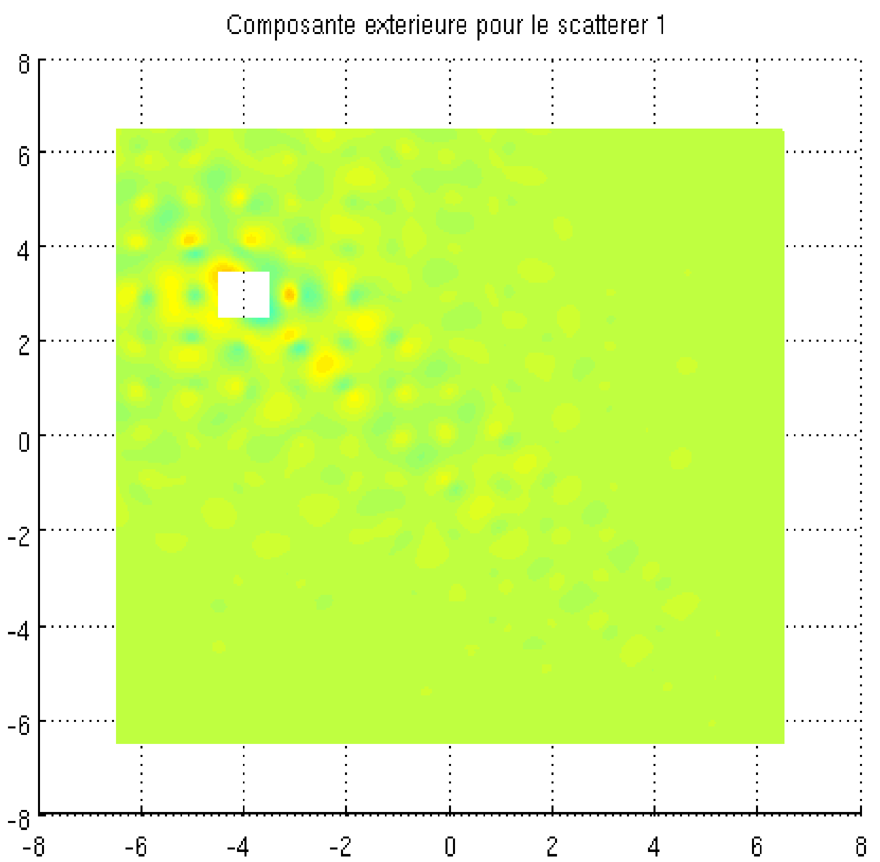
Numerical results

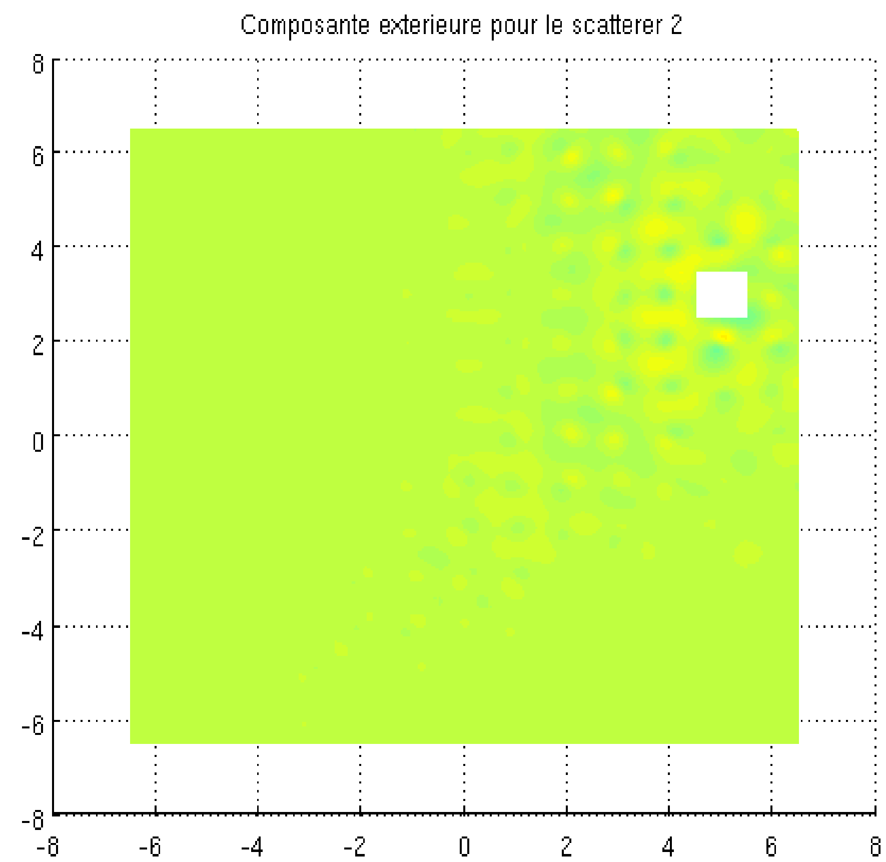


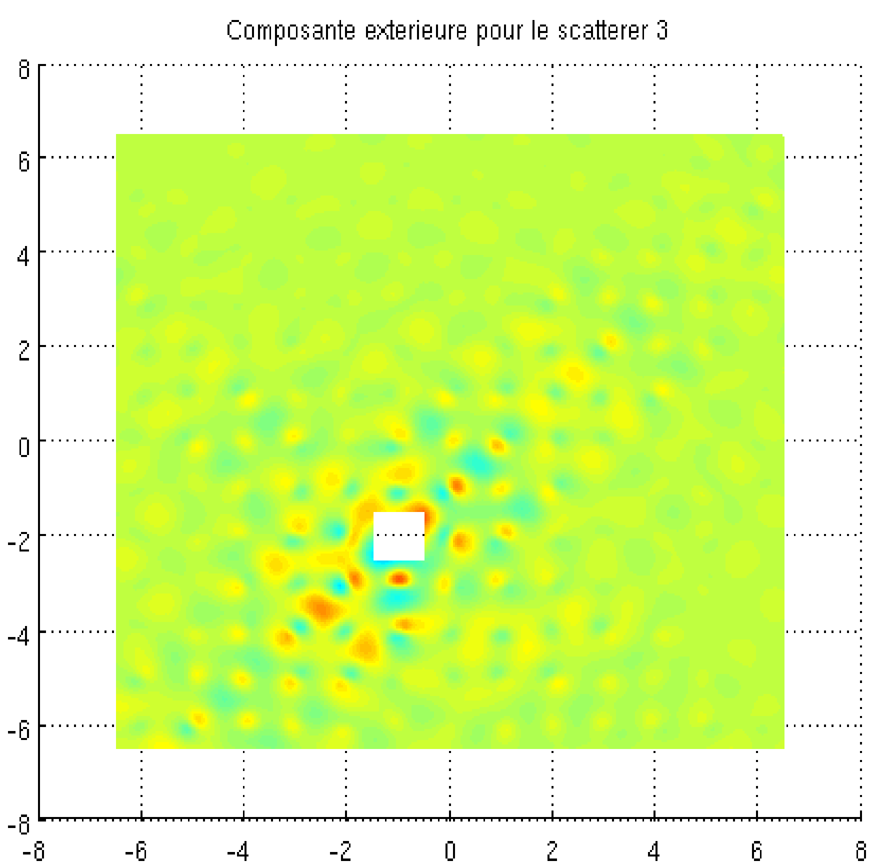
Numerical results

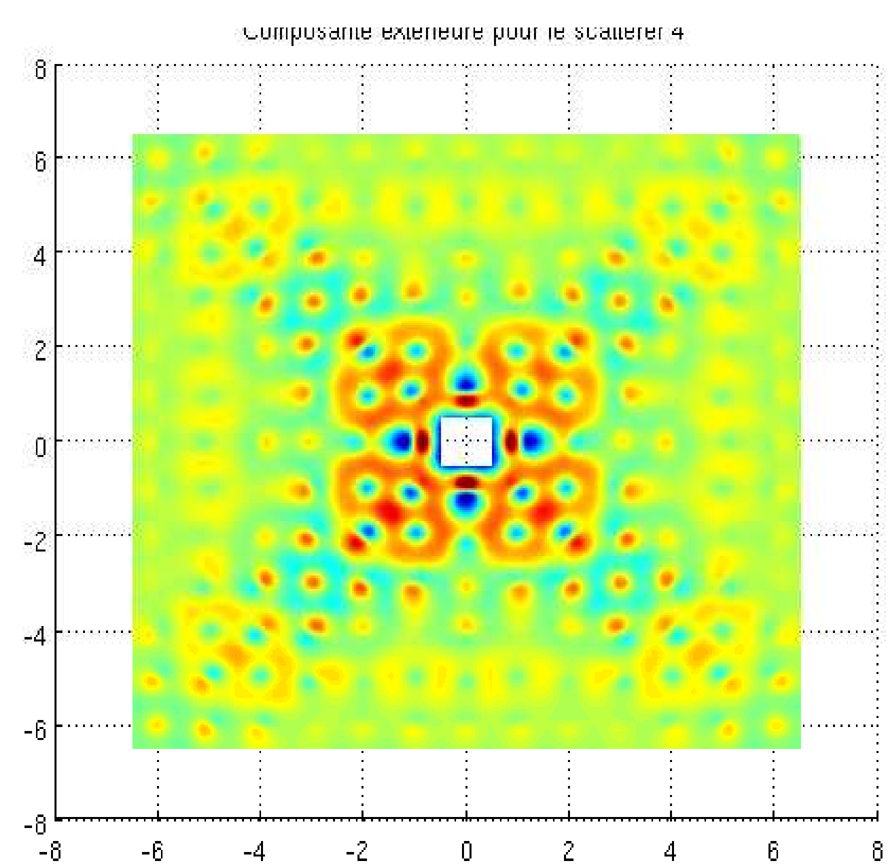


Numerical results

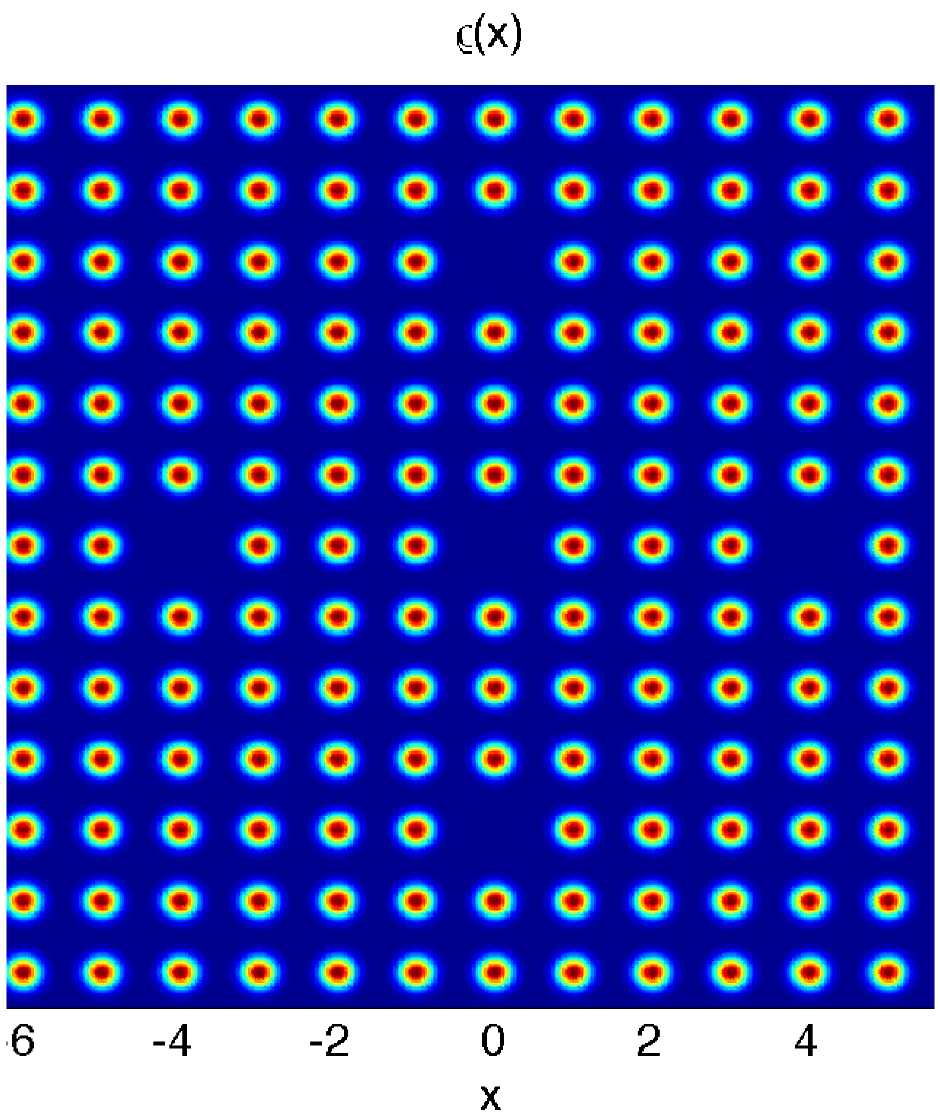


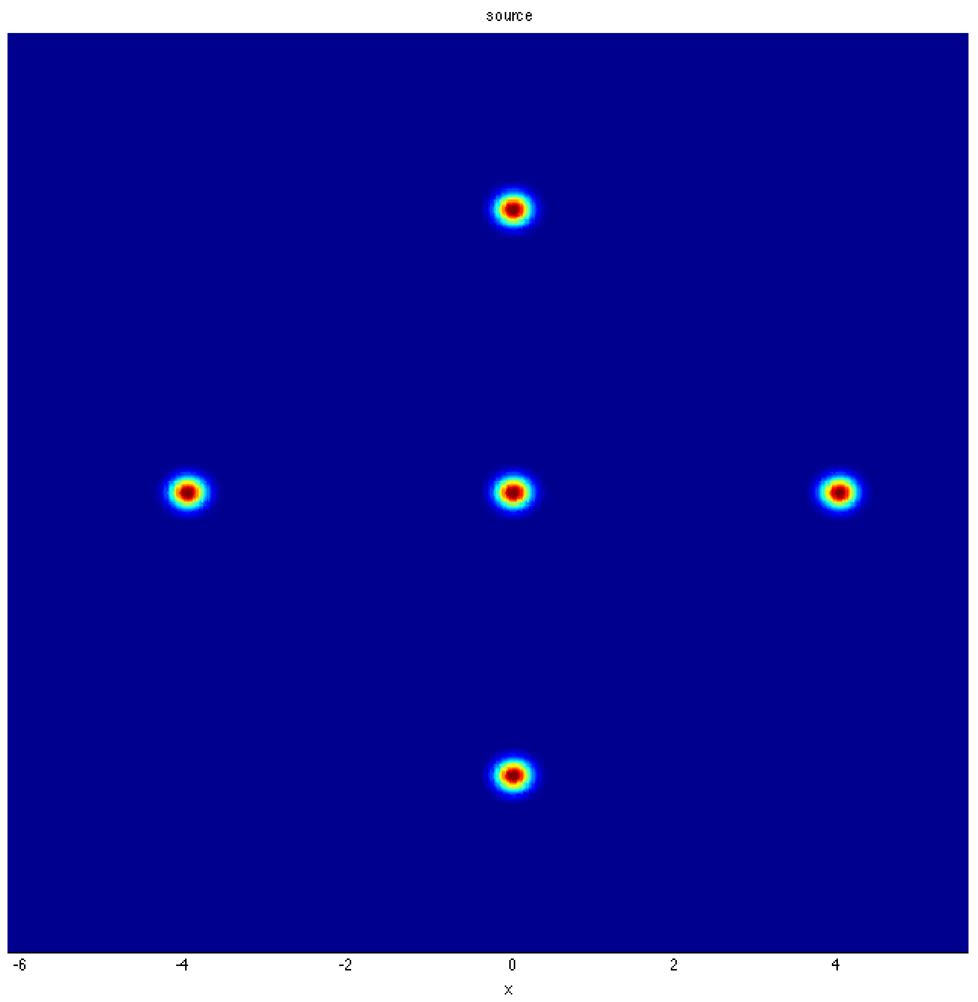


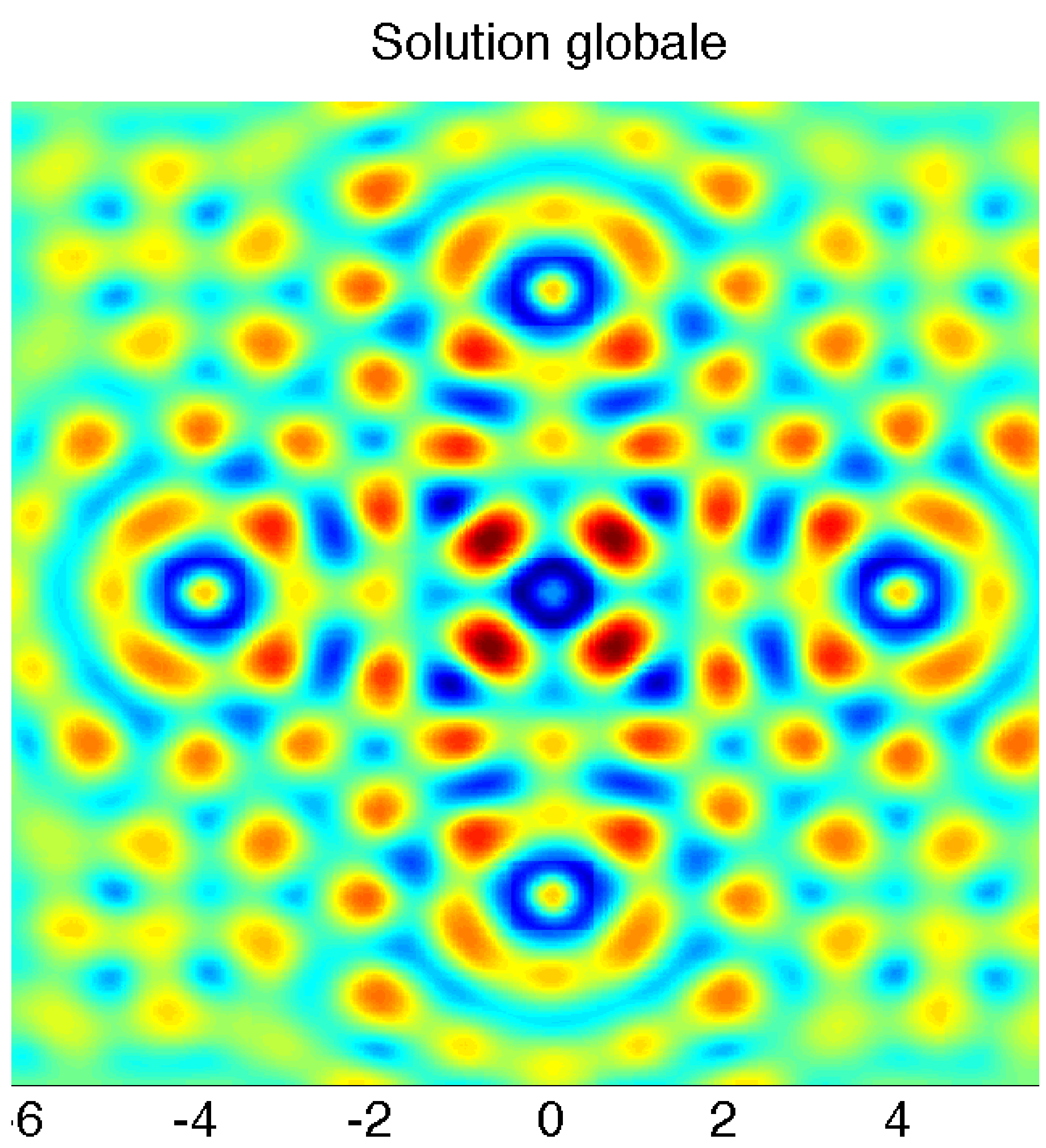




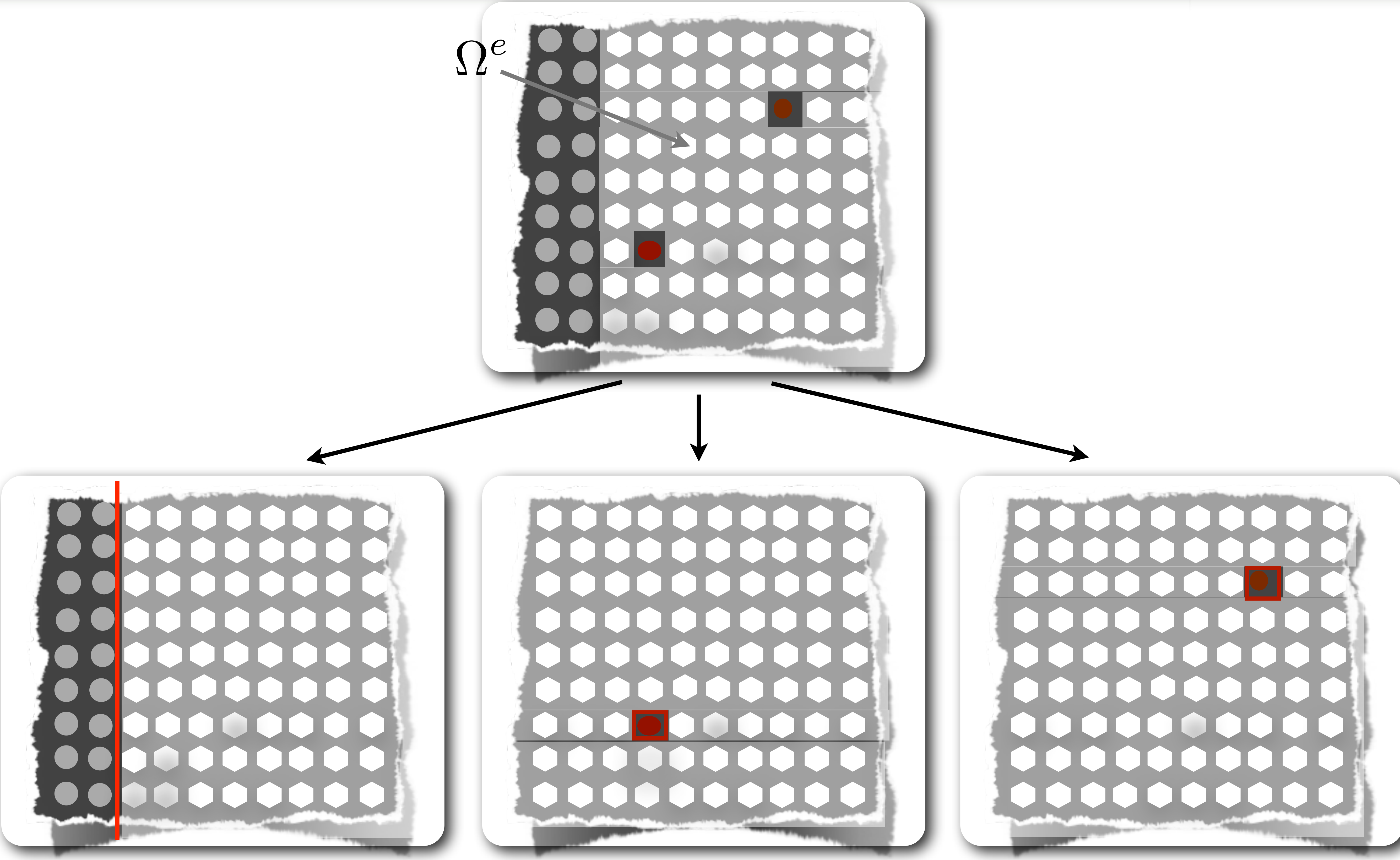
The 2D plane problem - Multiple scattering






Solution globale
-6
-4
-2
0
2
4

The 2D plane problem - Multiple scattering



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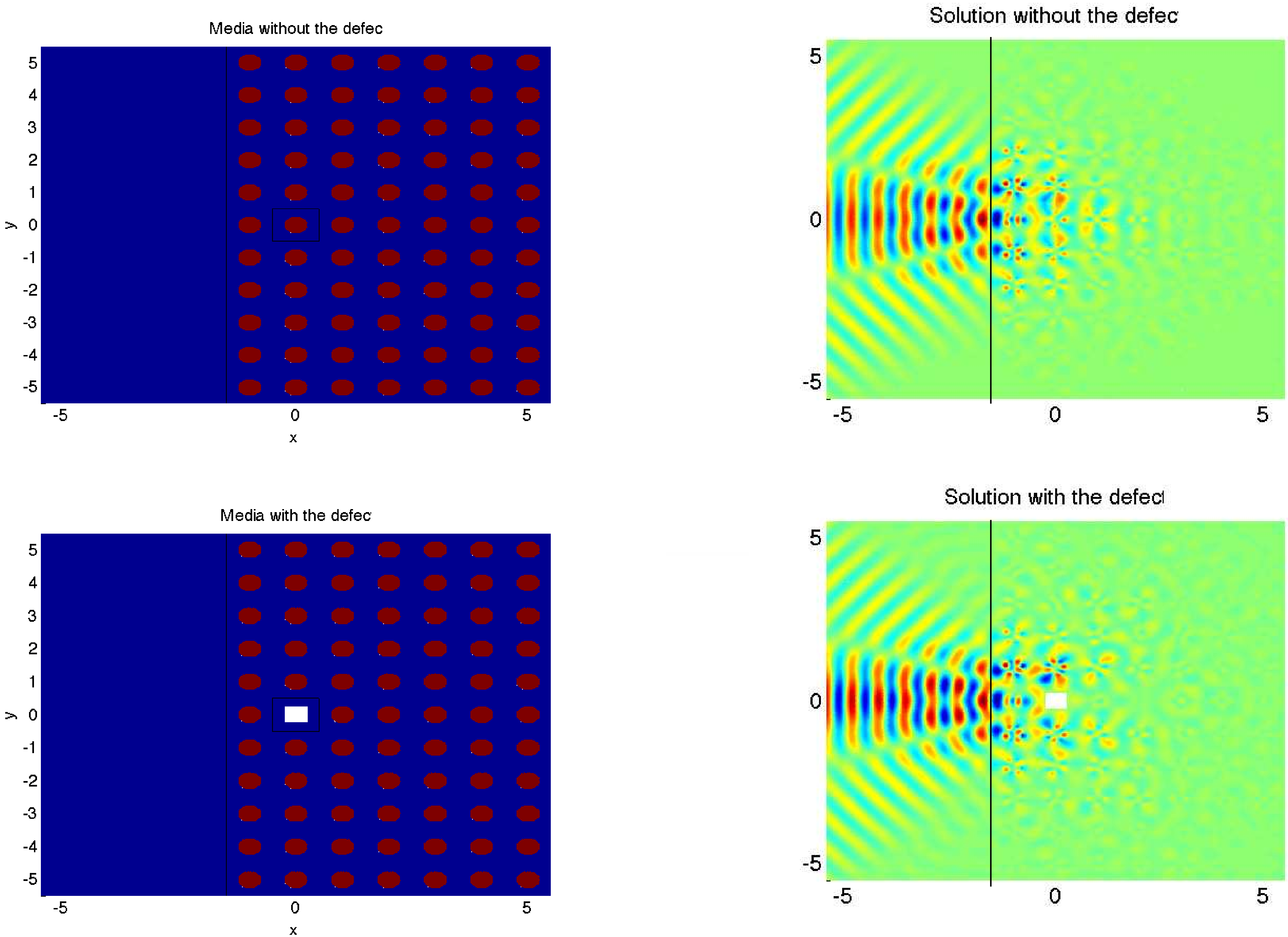
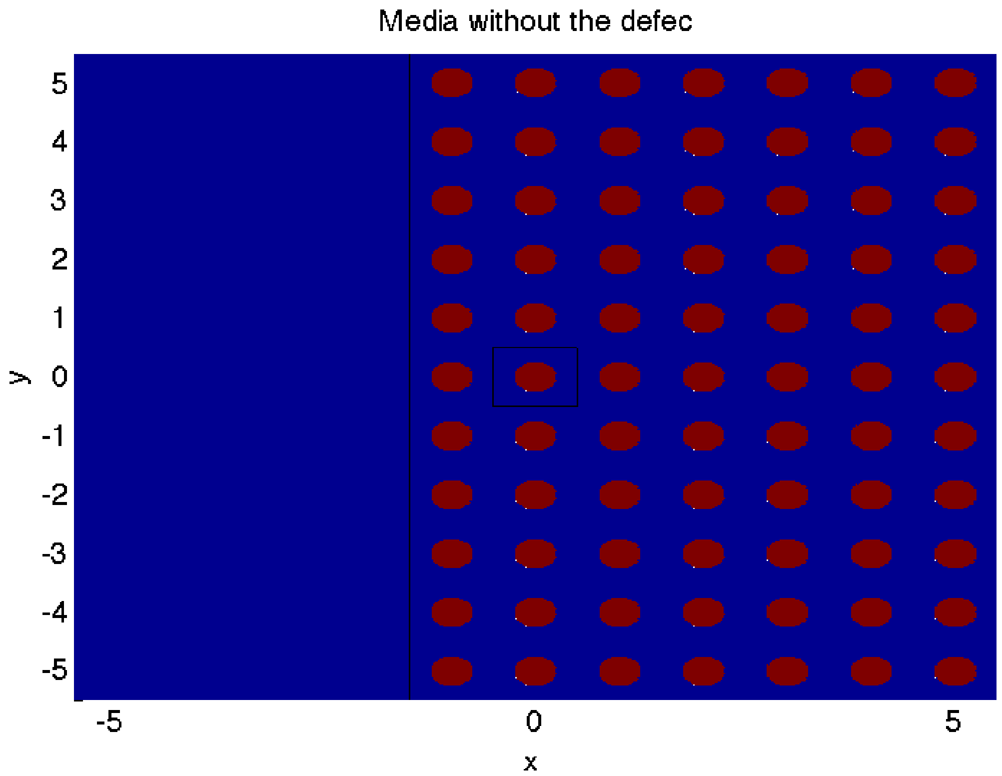
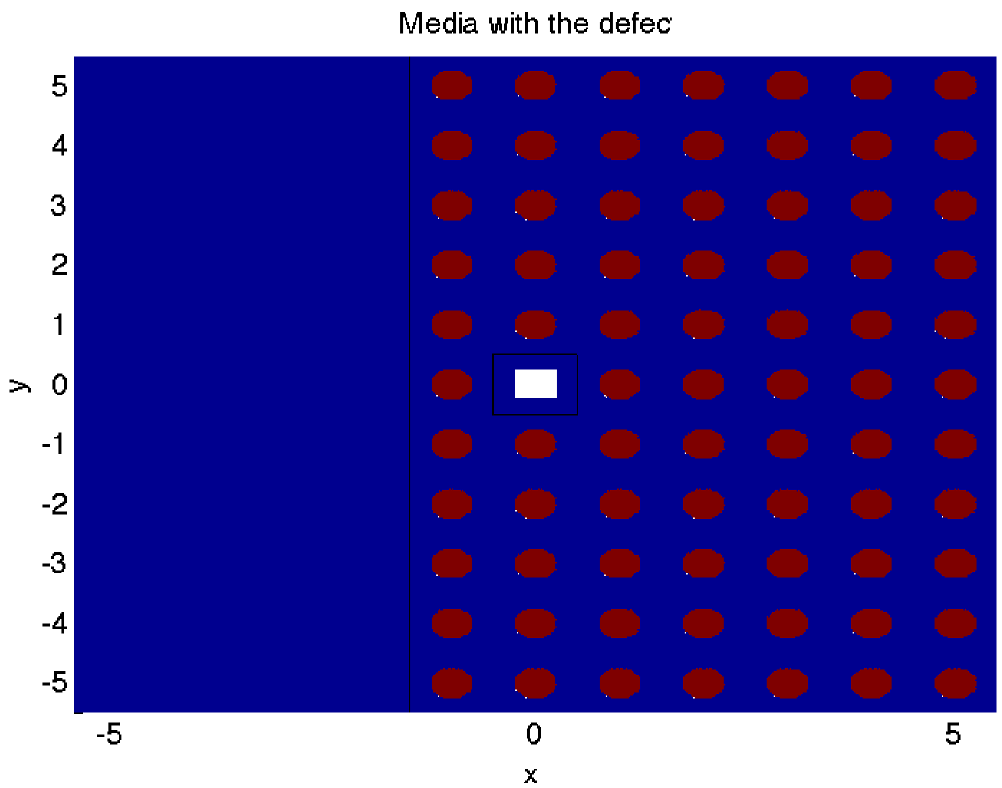
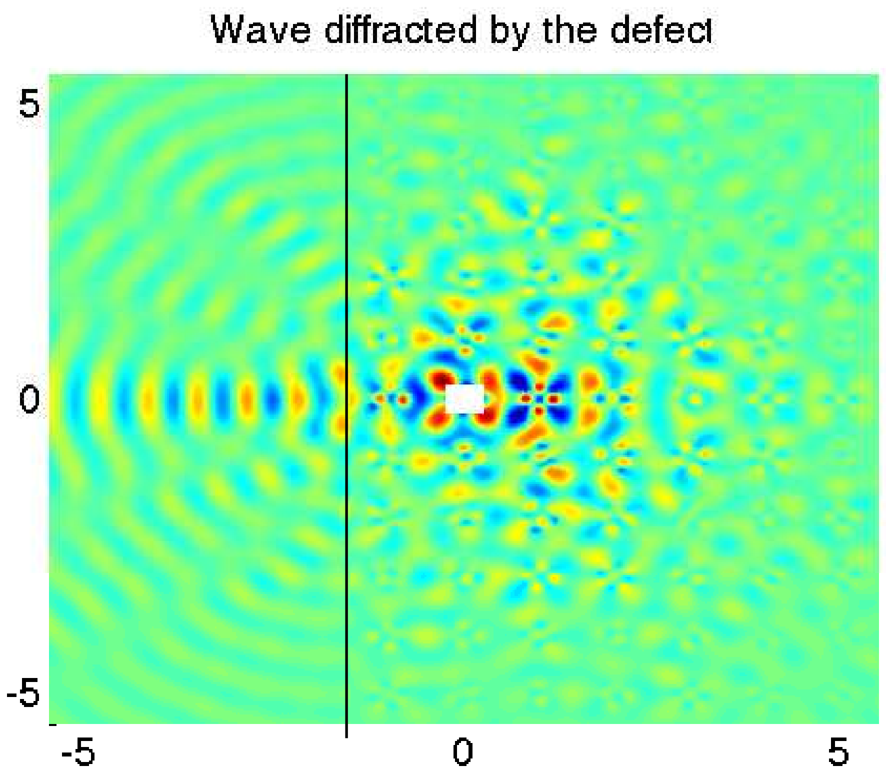


The 2D plane problem - Multiple scattering







Numerical methods for time harmonic scalar wave equation in locally perturbed periodic media - Part 4

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POEMS (UMR 7231 CNRS-INRIA-ENSTA)

Mainly based on joint works with Julien Coatleven, Patrick Joly and Jing-Rebecca Li

Numerical methods for time harmonic scalar wave equation in locally perturbed periodic media - Conclusions

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Conclusions for the 1D and waveguide problems

Absorbing media

The theory is complete and the numerical method works well

Non absorbing media

Limiting absorption principle for the definition of the DtN operators

Under some conditions on the periodic media, the problem is Fredholm

The problem is well posed except for a countable set of frequencies?

Conclusions for the 2D problems

Absorbing media

The theory is complete and the numerical method works well

The numerical analysis has to be done

Non absorbing media

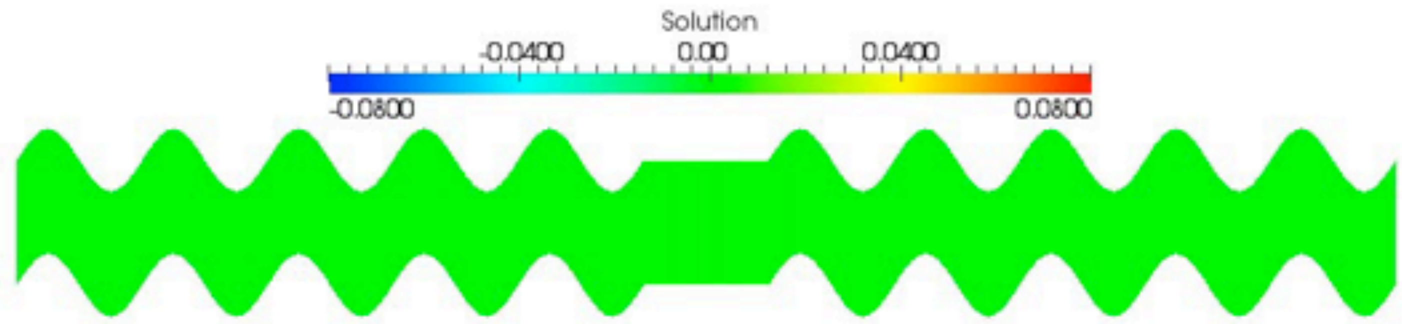
A numerical limiting absorption method is done for the other cases

The corresponding theory still raises challenging open questions

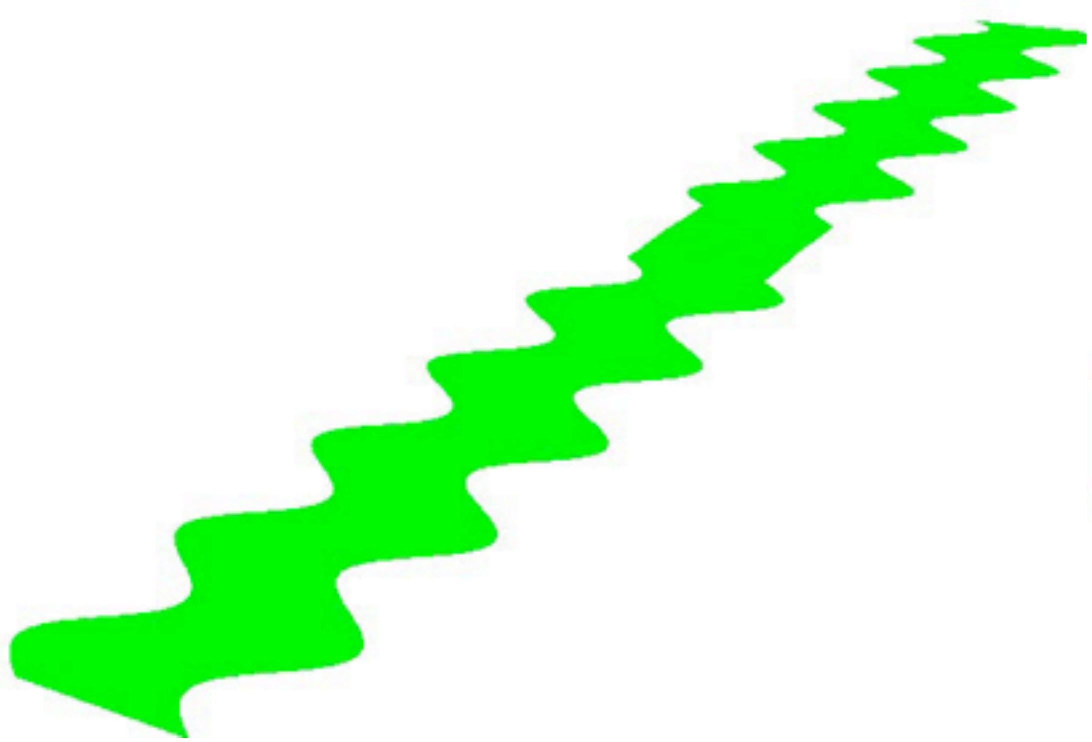
Numerical results for time domain problems

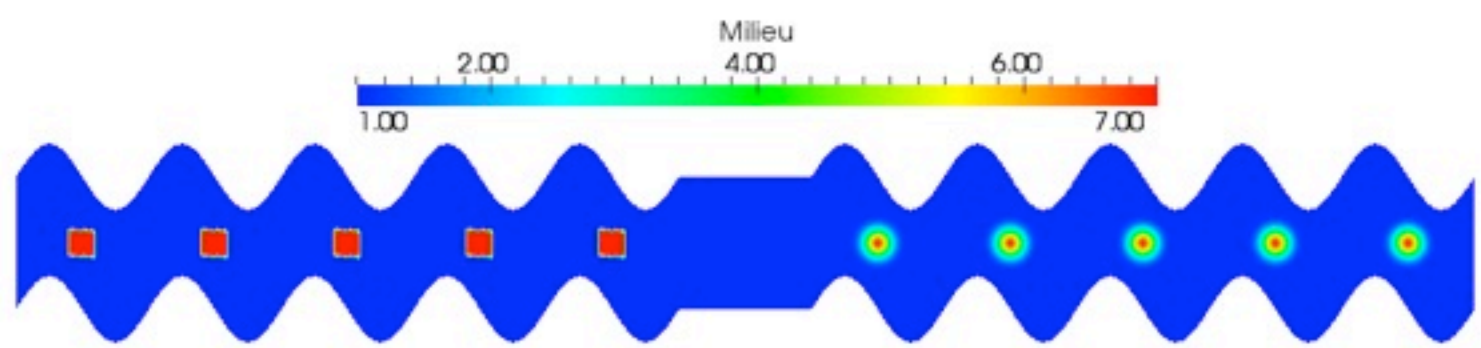
$$f = \left| \begin{array}{ll} A_f e^{-\left(\frac{(x-c_x)^2}{w_x^2}\right)} cos(\omega t) & \text{dans } \Omega_0 \\ 0 & \text{dans } \mathbb{R}^2 \setminus \Omega_0 \end{array} \right.$$

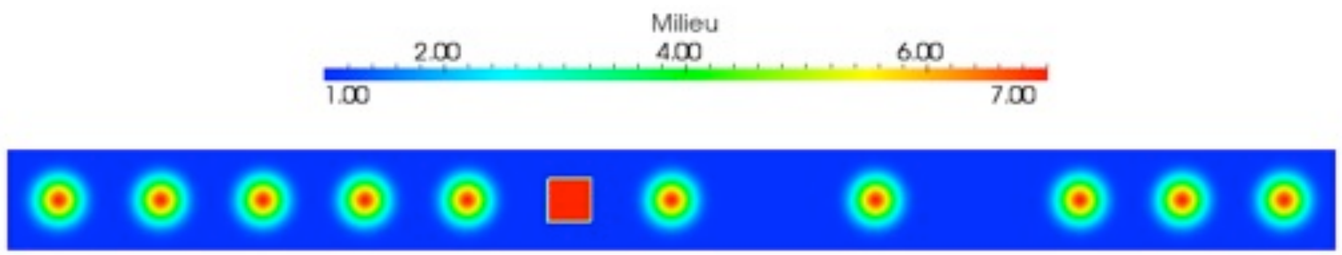
$$A_f = 3 \quad c_x = 0.5 \quad w_x = 0.2 \quad \omega = 5$$

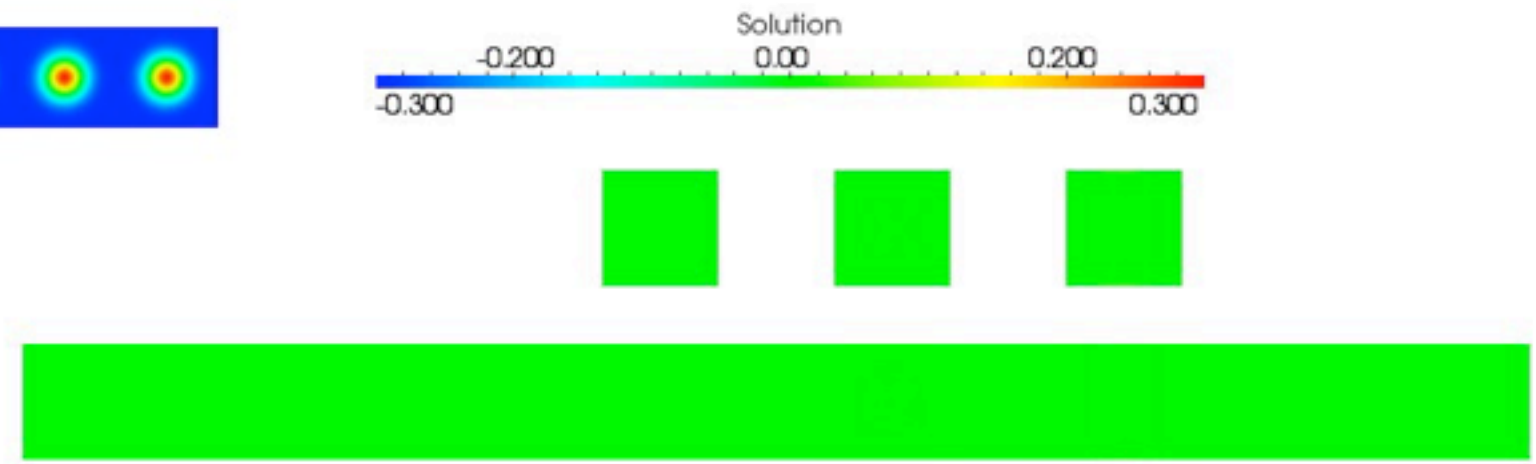


$$r = 2,\ \theta = 1/4,\ \Delta t = 0.04$$

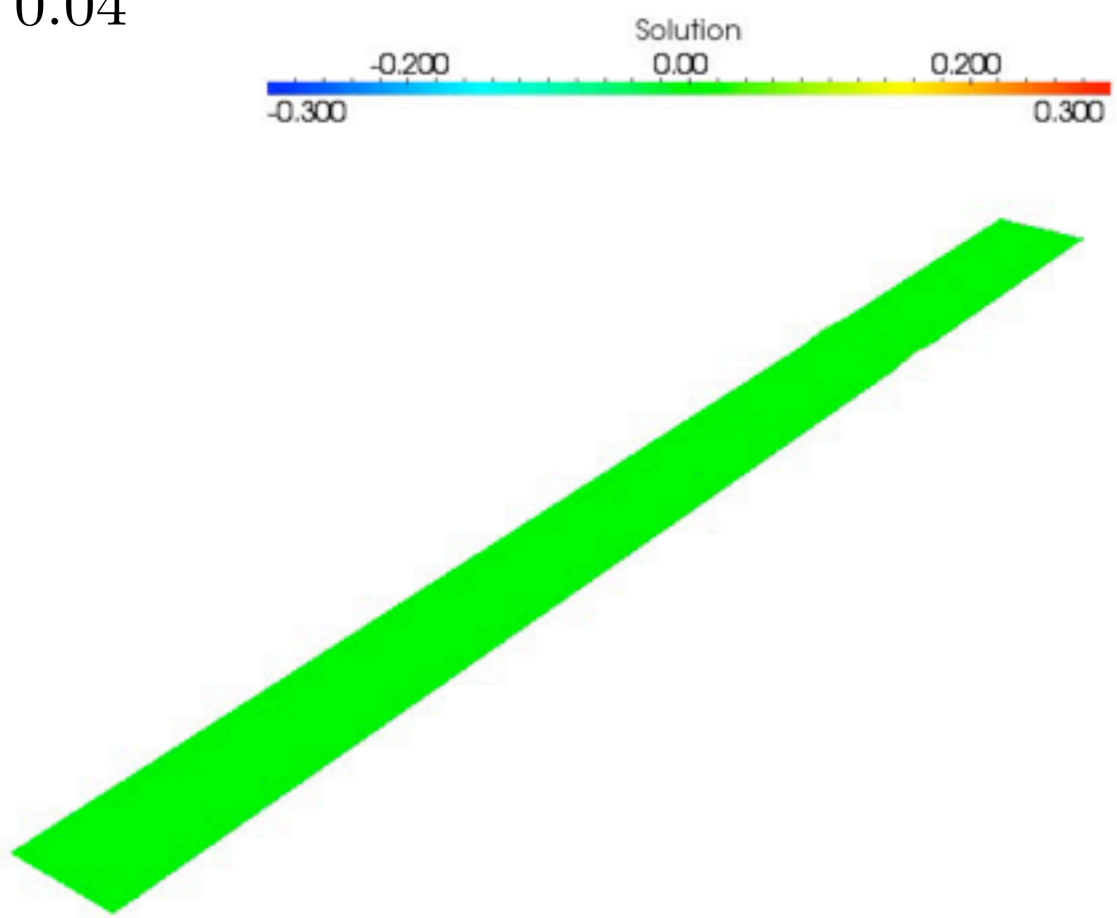




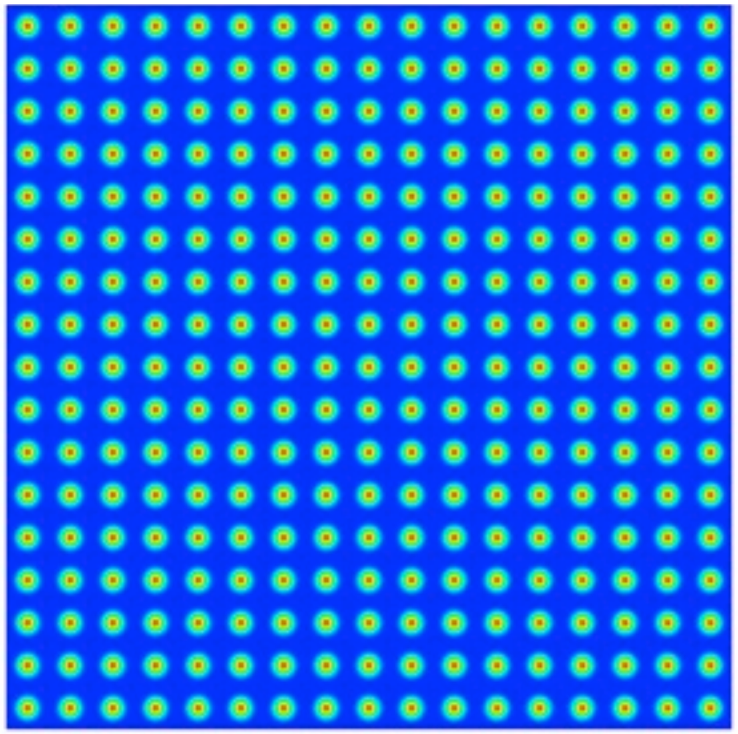


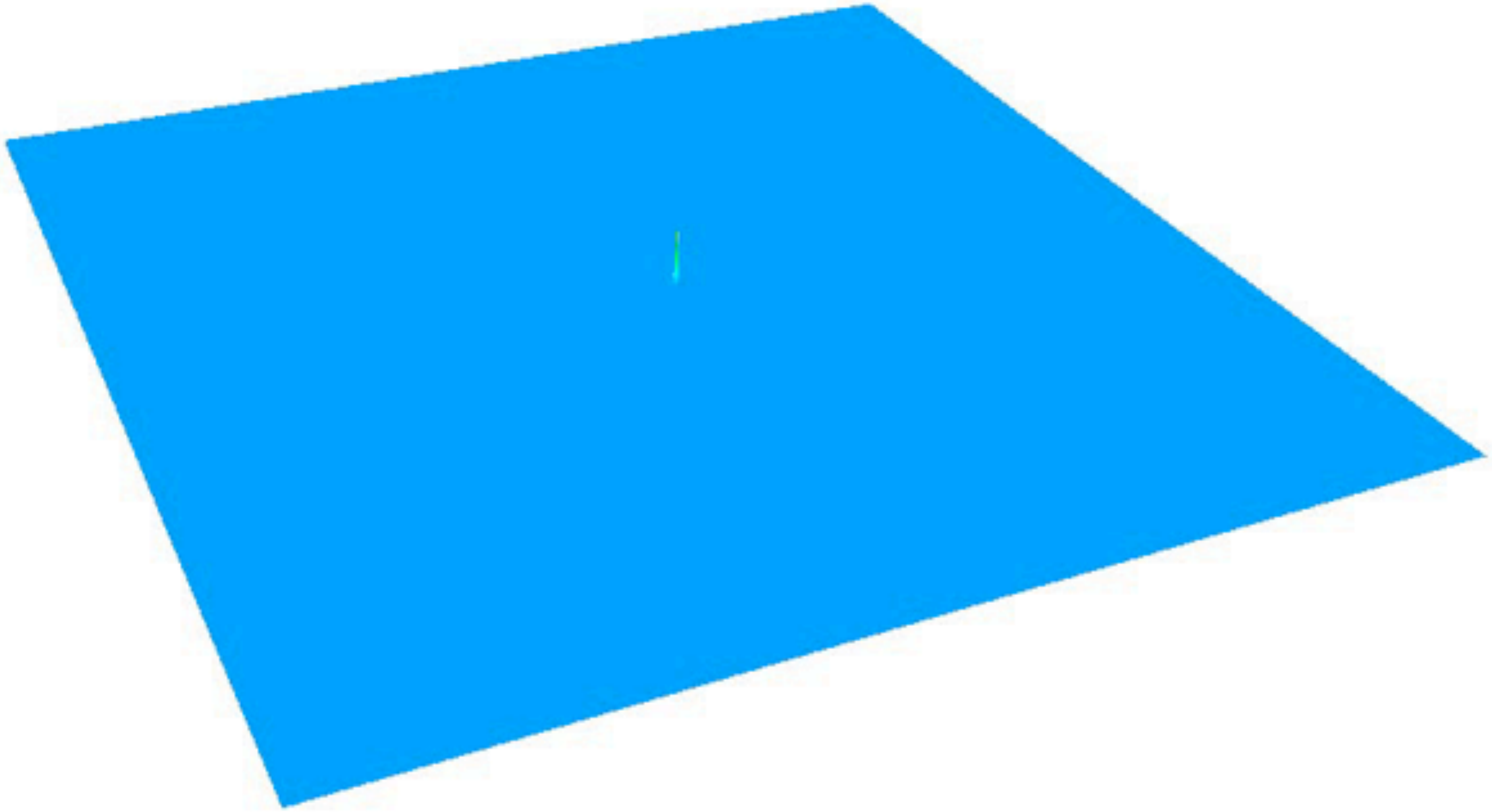


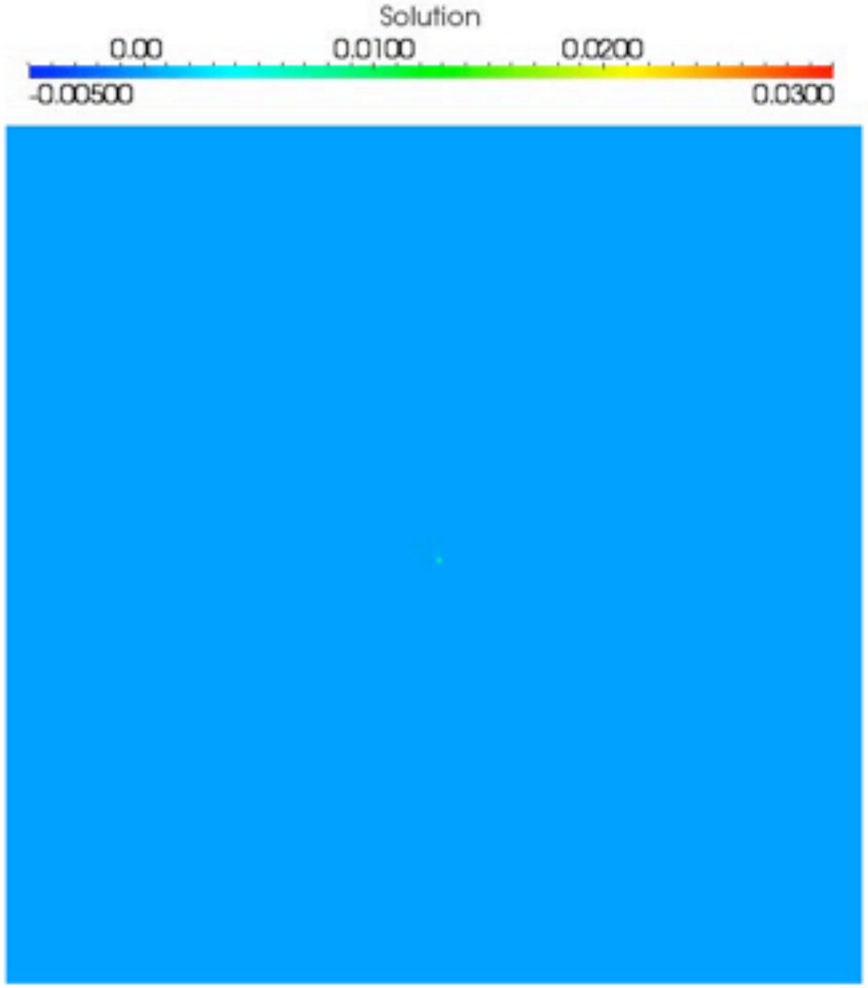
$r = 3,\ \theta = 1/4,\ \Delta t = 0.04$

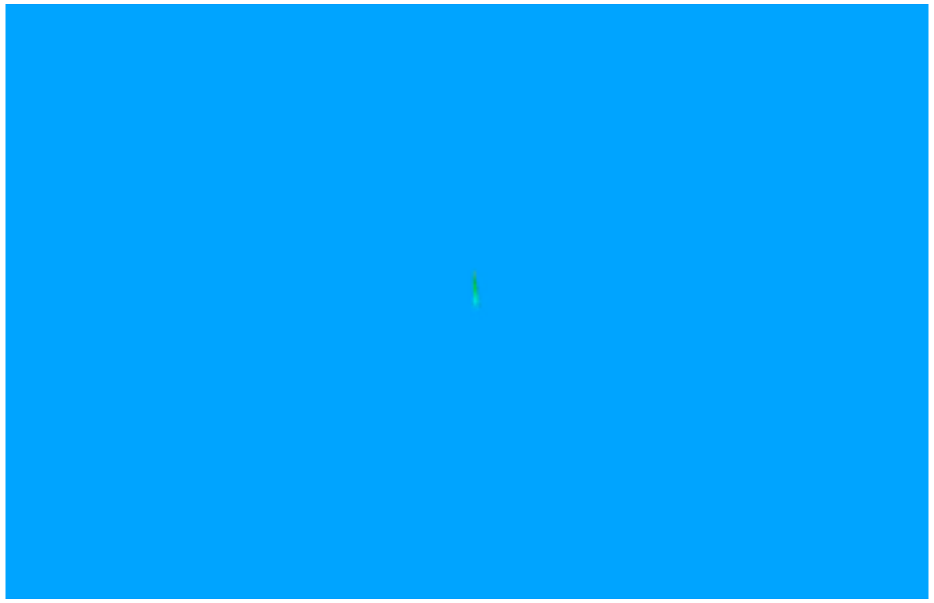


$r = 4,\ \theta = 1/4,\ \Delta t = 0.04$

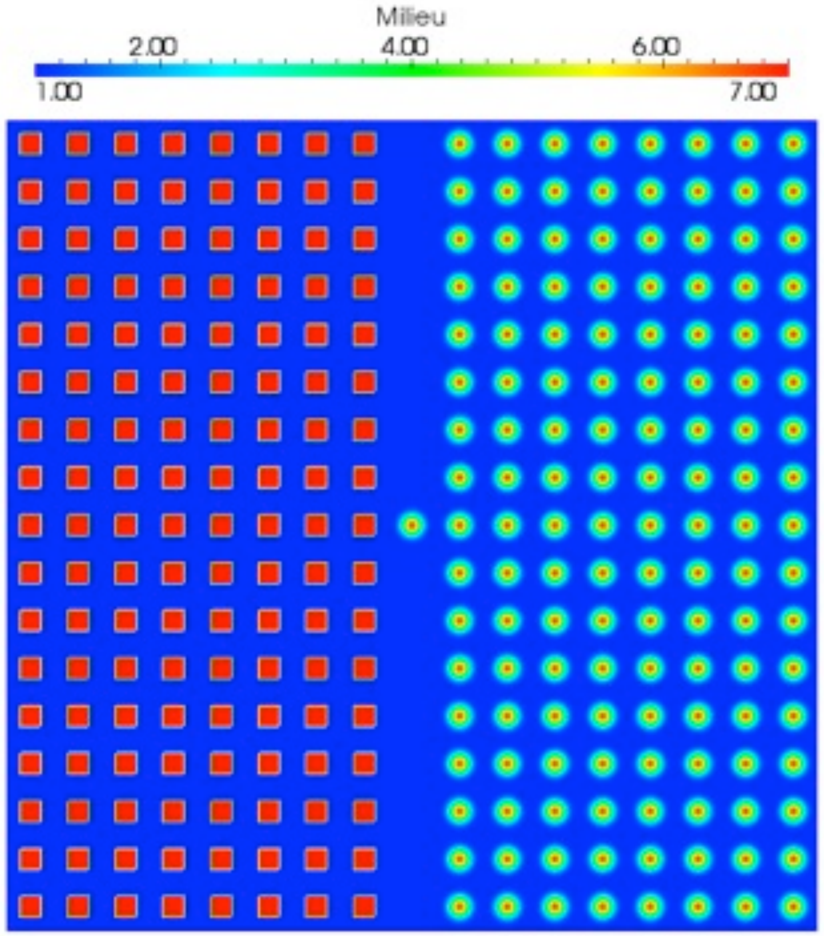


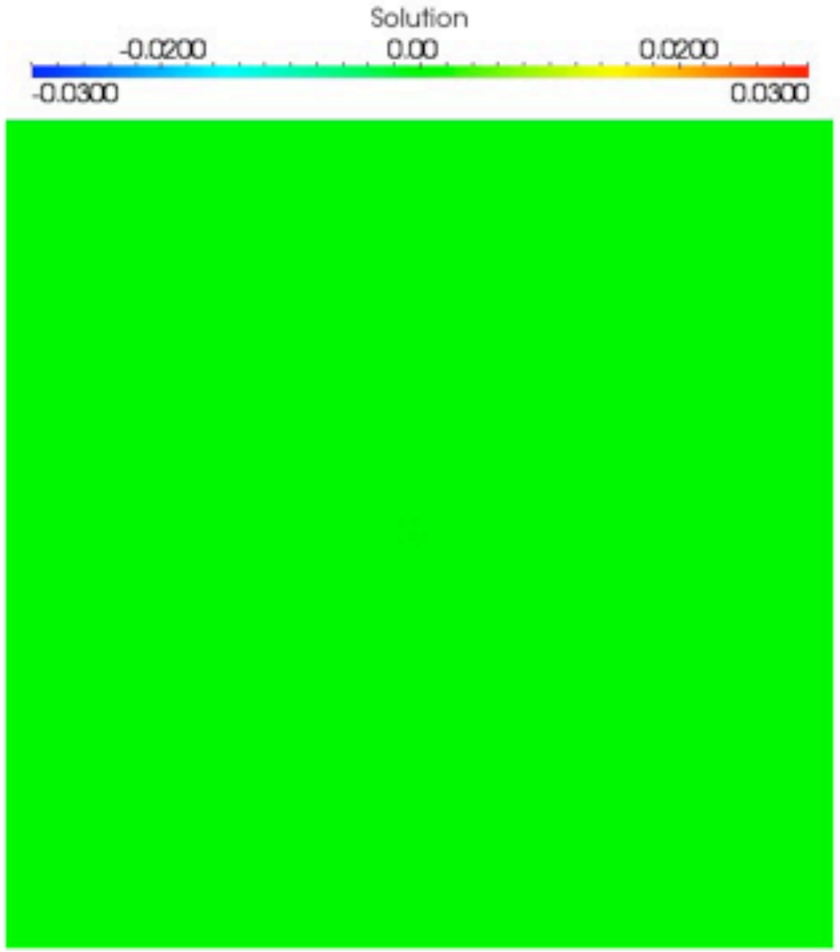


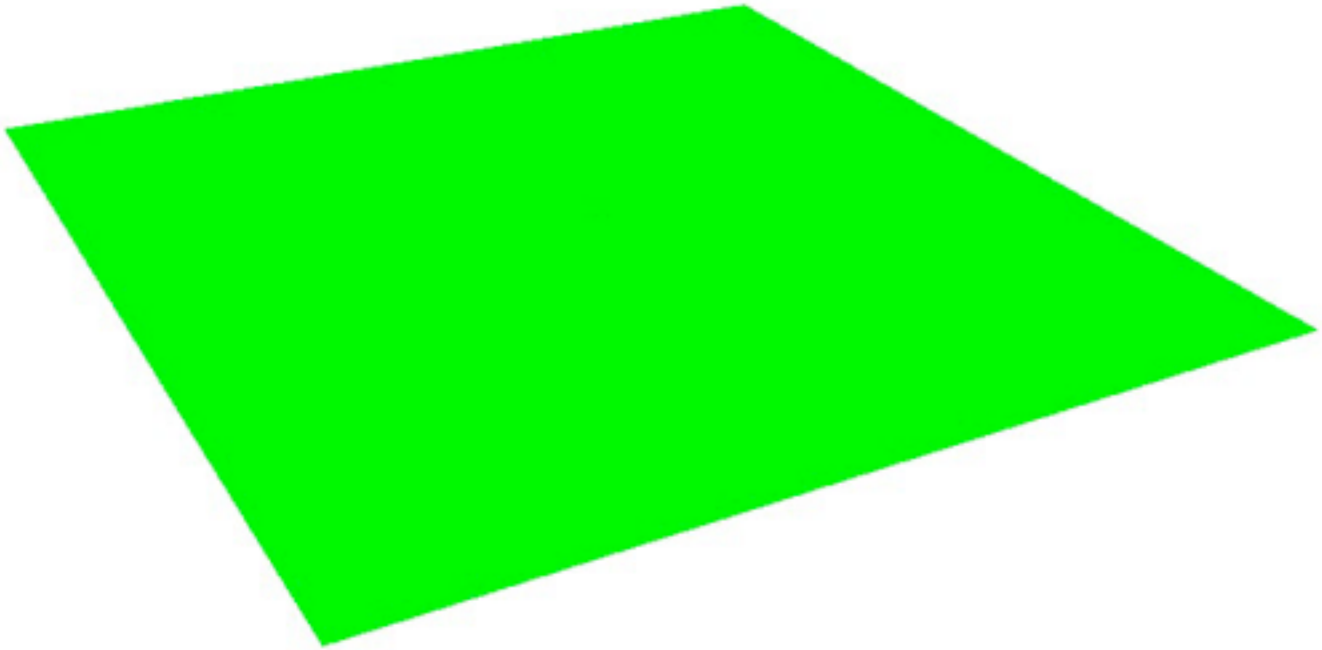




Numerical results for time domain problems







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