Identification of independent structural shocks in the presence of multiple Gaussian components

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Abstract

Several recently developed identification techniques for structural VAR models are based on the assumption of non-Gaussianity. So-called independence based identification provides unique structural shocks (up to scaling and ordering) under the assumption of at most one Gaussian component. While non-Gaussianity of certain interesting shocks, e.g., a monetary policy shock, appears rather natural, not all macroeconomic shocks in the system might show this clear difference from Gaussianity. We generalize identifiability by noting that even in the presence of multiple Gaussian shocks the non-Gaussian ones are still unique. Consequently, independence based identification allows to uniquely determine the (non-Gaussian) shocks of interest irrespective of the distribution of the remaining system. In an illustrative macroeconomic model the identified structural shocks confirm the results of previous studies on the early millennium slowdown. Furthermore, extending the time horizon provides full identification under the non-Gaussianity assumption.

Keywords: SVAR, identification, non-Gaussian, millennium slowdown.

JEL Classification: C32, E32.

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1 Introduction

Structural vector autoregressive (SVAR) models are frequently applied to identify the fundamental economic driving forces in macroeconomic systems. In this framework, diverse approaches aim at tracing macroeconomic variables back to orthogonal shocks (see Kilian and Lütkepohl, 2017, for an overview). While the identification procedures handle non-uniqueness of the structural matrix by building on certain statistical or economic assumptions, the views on the adequacy of these restrictions are diverging. Under Gaussianity, additional economic restrictions help to reduce the set of uncorrelated structural shocks, derived by any decomposition of the covariance matrix, to those in line with common economic beliefs (Sims, 1980; Blanchard and Quah, 1989; Faust, 1998; Uhlig, 2005). However, uncorrelated non-Gaussian structural shocks can still incorporate diverse forms of dependence. In order to separate the shocks and the associated responses completely, independent component analysis (ICA) methods uniquely identify the instantaneous response matrix for independent structural shocks under non-Gaussianity. These approaches base on the prominent theorem of Comon (1994) which indicates the existence of a unique structural matrix if the model contains at most one Gaussian structural shock (see, for instance, Moneta et al., 2013; Gouriéroux et al., 2017; Lanne et al., 2017).

When applying a structural VAR model the analyst is mostly interested in studying the responses to certain shocks only. For instance, the macroeconomic implications of monetary policy shocks have been widely analyzed by means of SVAR techniques. The distribution of the change in interest rates, estimated by a kernel density in Figure 1 (cf. Chiu et al., 2016), leads to the rather natural assumption that an unanticipated shock in monetary policy comes from a non-Gaussian distribution. However, different macroeconomic variables might be more 'balanced' in that they follow a distribution which is closer to Gaussianity (e.g. a supply or demand shock). In order to identify only parts of the system, we allow the K-dimensional vector of structural shocks ε_t to contain $1 < k_1 < K$ Gaussian components. In this setting, neither Gaussianity implies independence of all shocks nor ICA methods can just identify the whole system. We show that the $K - k_1$ non-Gaussian components of ε_t can still be uniquely identified by ICA methods. This result introduces flexibility by allowing for partial identification of the system after diagnosing (non-) Gaussianity of the structural shocks. Especially, when the effect of only certain structural shocks is of interest (and they are non-Gaussian), the distribution of the remaining system is irrelevant for their identification.

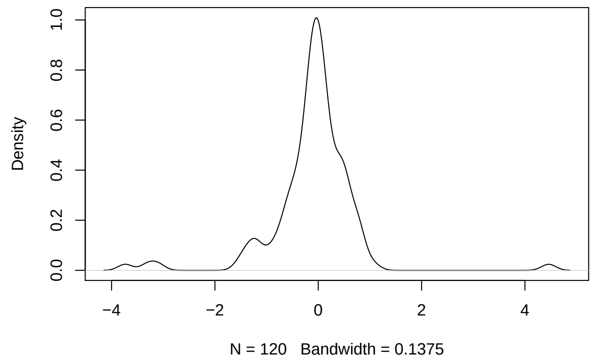

Figure 1: Kernel density estimate of the change in nominal interest rate in 1984–2002 (for a more detailed description of the data see Section 3).

We illustrate partial identification by re-investigating a four dimensional macroeconomic model in the spirit of Peersman (2005) who intended to identify the causes of the early millennium slowdown. More specifically, we identify two of four possible independent shocks by relying on a nonparametric dependence measure, the distance covariance. Studying quarterly data for 1980–2002, we interpret the identified oil price and monetary policy shocks in light of former replication studies. For an extended sample, more pronounced differences from Gaussianity arise. This allows full identification of the system and the interpretation of the response to all structural shocks.

In Section 2, we describe the model setting and the identification techniques for at most one and multiple Gaussian components. Section 3 contains the description and discussion of the estimation results for a four dimensional macroeconomic model. Section 4 concludes.

2 Model and identification

We consider a $K-$dimensional macroeconomic VAR model formulated as

$$\begin{aligned} y_t &= c_t + A_1 y_{t-1} + \ldots + A_p y_{t-p} + u_t, \\ &= c_t + A_1 y_{t-1} + \ldots + A_p y_{t-p} + \mathsf{B}\varepsilon_t = \mu + \sum_{i=0}^{\infty} \Phi_i \mathsf{B}\varepsilon_{t-i} \quad t = 1, \ldots, T, \end{aligned} \tag{1}$$

where c_t is a matrix of deterministic terms, y_t is $K \times 1$ dimensional and $A_1, \ldots, A_p$ and Φ_i are $K \times K$ matrices. For paraphrasing (1) we assume causality of the model, i.e., $\det \Phi(z) \neq 0$ for all $|z| \leq 1$ with $\Phi(z) = \sum_{i=0}^{\infty} \Phi_i z^i$ and $\Phi_0 = I_K$. Reduced form residuals correspond to error terms $u_t \sim (0, \Sigma_u)$ with non-singular covariance matrix $\Sigma_u = \mathsf{B}\mathsf{B}'$. The main interest of the following study is the identification of matrix B and the associated structural shocks $\varepsilon_t = \mathsf{B}^{-1}u_t$ with $E(\varepsilon_t) = 0$ and $\Sigma_\varepsilon = \mathsf{B}^{-1}\Sigma_u\mathsf{B}^{-1} = I_K$. For this purpose, the literature on SVAR models incorporates numerous approaches to identify the non-unique factor B properly relying on either statistical or economic a-priori assumptions (for a textbook treatment of SVARs see Kilian and Lütkepohl, 2017).

2.1 Independence based identification

Recently developed statistical identification procedures exploit the non-normality of structural shocks building on results from independent component analysis (Moneta et al., 2013; Lanne et al., 2017; Gouriéroux et al., 2017). For the vector of reduced form errors $u_t \in \mathbb{R}^K$, ICA aims at determining the so-called mixing matrix B for which the components of $\mathsf{B}^{-1}u_t = \varepsilon_t$ are independent. Following the fundamental result of Comon (1994), ICA uniquely identifies matrix B up to column signs and ordering by allowing the vector of independently distributed structural shocks ε_t to contain at most one Gaussian component $\varepsilon_{t,k}$.

In the following, we describe identification in the case of one and multiple Gaussian components on the basis of an ICA procedure adapted from Matteson and Tsay (2017). The distance covariance, a nonparametric dependence measure introduced in Székely et al. (2007), is applied to determine least dependent shocks and thereby, to identify the associated matrix B. It might be noteworthy that similar ICA-based identification procedures lead to the same theoretical results in the case of multiple Gaussian components.

2.1.1 Identification with at most one Gaussian structural shock

Moneta et al. (2013) have adopted ICA to determine optimal variable orderings in recursive systems of non-Gaussian structural shocks. However, the a-priori focus on triangular schemes appears restrictive in an economic context. Determining the underlying distribution family a-priori, Lanne et al. (2017) apply ML estimation to determine the matrix B. Moreover, nonparametric dependence measures provide an alternative tool for identification avoiding any restrictive assumption on the distribution of ε_t. In this work, we rely on the so-called distance covariance of Székely et al. (2007) applied in the course of ICA by Matteson and Tsay (2017).[1] The set of possible decompositions

[1] Diverse alternative criteria have been studied in preliminary analyses (avalaible on request) where especially the Cramér-von Mises distance turns out as a robust alternative to measure dependence nonparametrically.

of the least squares covariance estimator $\mathsf{B}(\theta) = DQ(\theta)$ is defined with respect to Choleski factor D and the vector of rotation angles θ of the Givens matrices $Q(\theta)$. We estimate the covariance matrix once by least squares and different decompositions evolve by drawing from the set of all rotation angles θ. Accordingly, the distance covariance $\mathcal{U}_T(\hat{\varepsilon}_t(\theta))$ can be calculated from $\hat{\varepsilon}_t(\theta) = \mathsf{B}(\theta)^{-1}\hat{u}_t$ where $\hat{u}_t$ are the least squares residuals. Minimization of the distance covariance $\hat{\theta} = \mathrm{argmin}_\theta \mathcal{U}_T(\hat{\varepsilon}_t(\theta))$ consequently determines the estimated matrix $\widehat{\mathsf{B}} = \mathsf{B}(\hat{\theta})$ and the associated least dependent shocks $\hat{\varepsilon}_t(\hat{\theta})$. For details on the exact minimization procedure and the empirical definition of the dependence measure we refer to Matteson and Tsay (2017). In this study, we apply the function *steadyICA* from the `R` package `steadyICA` (Risk et al., 2015) to determine $Q(\hat{\theta})$ and thus, $\widehat{\mathsf{B}}_{\mathrm{dCov}} = \mathsf{B}(\hat{\theta})$.

2.1.2 Identification with multiple Gaussian structural shocks

More generally, let the vector ε_t contain $1 \leq k_1 \leq K$ Gaussian random variables. If the number of Gaussian components exceeds one, i.e. $k_1 > 1$, matrix B can no longer be uniquely identified and consequently, the structural shocks $\varepsilon_t = \mathsf{B}^{-1}u_t$ can not be separated by means of ICA. However, by an intuitive generalization of Comon's theorem the $K - k_1$ non-Gaussian components of ε_t remain unique. We formulate this result in the following proposition for two random vectors ε_1, $\varepsilon_2 \in \mathbb{R}^K$, representative for vectors with independent components not distinguishable by means of ICA. Within these vectors the Gaussian components are ordered first.

Proposition 1. *Let ε_1 be a vector with independent components of which only w.l.o.g. the first k_1 components are Gaussian. Let C be an orthogonal $K \times K$ matrix and $\varepsilon_2 = C\varepsilon_1$ such that the first k_1 entries of ε_2 are Gaussian. The components of ε_2 are mutually independent if and only if $C = \begin{pmatrix} Q & 0 \\ 0 & \Lambda P \end{pmatrix}$ where matrix Q is an orthogonal $k_1 \times k_1$ matrix, Λ is a $(K - k_1) \times (K - k_1)$ diagonal matrix and P is a permutation matrix.*

The proof is given in the Appendix and represents an alternative to Boscolo et al. (2002). For matrix C as defined in Proposition 1, ICA can not distinguish between $\mathsf{B}C$ and B, in other words $\varepsilon_t = (\mathsf{B}C)^{-1}u_t$ also comprises independent components. In the following, we apply the ICA procedure of Matteson and Tsay (2017) to models with several Gaussian structural shocks. Statistical properties, as consistency, of the *steadyICA* algorithm under multiple Gaussian components transfer to the subsample of non-Gaussian variables. Leaving the formal derivation aside we assume that the first k_1 columns of $\widehat{\mathsf{B}}_{\mathrm{dCov}}$ (if Gaussian components are ordered first) are not uniquely determined as the Gaussian variables can not be distinguished (Hyvärinen et al., 2001). In contrast, the remaining $K - k_1$ columns of $\widehat{\mathsf{B}}_{\text{dCov}}$ are unique. Along these lines, for at most one Gaussian component all columns of $\widehat{\mathsf{B}}_{\text{dCov}}$ are unique. For applicability of the identification technique it is essential to decide on the number of Gaussian components first.

Decide on the number of Gaussian components

Various alternative uni- and multivariate tests for normality are present in the literature. A selection of tests is, for instance, implemented in the `R` package `normtest` (Gavrilov and Pusev, 2015). Moreover, diverse strategies can be pursued to assess normality of a multivariate vector of structural shocks $\hat{\varepsilon}_t$. In the following, we choose two alternative approaches. First, we test separately on Gaussianity of the components and secondly, we apply a test which decides on the number of non-Gaussian components in ICA. The results of separate univariate Jarque-Bera (JB) tests provide evidence for Gaussianity of the structural shocks determined by independence based identification, e.g. $\hat{\varepsilon}_t = \widehat{\mathsf{B}}_{\text{dCov}}^{-1}\hat{u}_t$. Note that the results from alternative univariate tests provide similar test outcomes and are not displayed here. Under the null hypothesis of the JB test the shock exhibits a Gaussian distribution. Thus, if the null hypothesis is rejected we assume that the associated shock can be uniquely identified by means of ICA.

However, the estimated structural shocks $\hat{\varepsilon}_t$ and their distribution might depend on the underlying identification procedure. To evaluate robustness of the JB test decisions, we apply techniques based on fourth order blind identification (FOBI) which have evolved in the course of non-Gaussian component analysis (NGCA) to isolate non-Gaussian from Gaussian components. In their `R` package *ICtest*, Nordhausen et al. (2016) have implemented several tests to decide on the number of non-Gaussian, so-called interesting, components within a set of variables. We apply the version implemented in the function `FOBIboot` which uses a bootstrap procedure. The test applies FOBI to trace the vector of reduced form residuals back to Gaussian and non-Gaussian sources. The corresponding null hypothesis states that there are k_1 Gaussian components and $K - k_1$ non-Gaussian components. For further details on the test and the implementation we refer to the manual of the `R` package (Nordhausen et al., 2016).

It might be noteworthy that the JB tests on Gaussianity of the structural shocks and the application of one overall test for Gaussian components provide a test decision derived under different significance levels. Either four separate tests on a certain level are performed or one single test helps, for instance, to decide about two Gaussian components on one level. We apply and compare both approaches in the subsequent application to a four dimensional macroeconomic model.

3 Reassessing causes of the early millennium slowdown

We consider the model in (1) where now $y_t = (\Delta oil_t, \Delta y_t, \Delta p_t, s_t)$ contains first differences of oil prices Δoil_t, output growth Δy_t, consumer inflation Δp_t and the short term interest rate s_t. Peersman (2005) applies this model setting to study the causes of the early millennium slowdown in 2001. In the following, we will consider the model in two variations of the sample period. First, we replicate the study of Peersman (2005) for the original sample 1980Q1–2002Q2. An extended sample includes data until 2007Q4 to further assess causes of the slowdown in 2001.[2] For the two samples we examine applicability of independence based identification by assessing Gaussianity of the shocks. Furthermore, we analyze the impulse responses estimated by means of the technique which relies on the distance covariance.

	$\hat{\varepsilon}_1$	$\hat{\varepsilon}_2$	$\hat{\varepsilon}_3$	$\hat{\varepsilon}_4$
JB	56.225	1.045	0.060	23.686
p-value	0.000	0.527	0.969	0.005

H_0 :	$k_1 = 2$	$k_1 = 3$
Test Stat.	16.312	491.88
p-value	0.915	0.035

Table 1: JB test results for $\hat{\varepsilon}_t = \widehat{\mathsf{B}}_{\text{dCov}}^{-1}\hat{u}_t$ for sample 1980Q1–2002Q2 (left-hand side table). Tests on non-Gaussian components in $\hat{u}_t$: we can reject that there are $k_1 = 3$ Gaussian components but we can not reject that there are $k_1 = 2$ Gaussian components at a reasonable significance level.

Table 1 and 2 display the outcome of separate JB tests for the structural shocks $\hat{\varepsilon}_t = \widehat{\mathsf{B}}_{\text{dCov}}^{-1}\hat{u}_t$ and sample periods 1980Q1–2002Q2 and 1980Q1–2007Q4, respectively.[3] Alongside, we display statistics and p-values of the tests on interesting, i.e. non-Gaussian, components. The JB test results hint at the presence of two Gaussian components ε_2 and ε_3 on the shorter horizon (Table 1). In the larger sample we reject normality of three of the four components at 10% significance level based on the JB tests (Table 2). By means of the test on interesting components we obtain

[2] It might be noteworthy that Peersman (2005) studies data for the US, the Euro area and the industrialized world. He argues that the effects appear the most pronounced in the US. As noted by Grant (2015) differences between the results and Peersman (2005) may occur due to data deviations.

[3] Note that slight differences to exact p-values might be caused by the Monte Carlo simulation used for calculation of the test distribution. However, we assume that the main conclusions remain unchanged.

the same result in the smaller sample. However, relying on this test, we might still assume the presence of two Gaussian components in the larger sample.

	$\hat{\varepsilon}_1$	$\hat{\varepsilon}_2$	$\hat{\varepsilon}_3$	$\hat{\varepsilon}_4$
JB	63.272	3.623	0.476	70.257
p-value	0.000	0.099	0.76	0.000

H_0 :	$k_1 = 2$	$k_1 = 3$
Test Stat.	69.288	1369
p-value	0.602	0.005

Table 2: JB test results for $\hat{\varepsilon}_t = \widehat{\mathsf{B}}_{\text{dCov}}^{-1}\hat{u}_t$ for sample 1980Q1–2007Q4. Tests on non-Gaussian components in $\hat{u}_t$: we can reject that there are $k_1 = 3$ Gaussian components but we can not reject that there are $k_1 = 2$ Gaussian components at 10% significance level.

Following Section 2.1.2, we assume that the distance covariance uniquely identifies the non-Gaussian shocks in the smaller and all shocks in the larger sample (relying on the JB test results). Further differences caused by the sample choice are reflected in the impulse responses in Figure 2 calculated using independence based identification. The displayed confidence intervals are calculated from a wild bootstrap procedure as, for instance, described in Herwartz and Plödt (2016). First, we notice that the confidence intervals in the shorter sample are mostly wider. This seems an intuitive consequence of the larger and more likely identified (because of non-Gaussianity) model exhibiting smaller estimation uncertainty. Furthermore, the point estimates of the dynamic responses are partly shifted which we attribute to a change in the data (i.e. the relations between variables) as well as the adequacy of the identification approach. However, in both cases we obtain two uniquely identified shocks, the first and the fourth, and can observe that the corresponding impulse responses appear very similar in both samples. Based on the reasoning of the following paragraph we label the first shock an oil price and the fourth a monetary policy shock. For these derivations we proceed with the model including data up to 2007Q4, merely to overcome the identification issues. However, it might be noteworthy that the results for the oil price and the monetary policy shock hold similar in the smaller sample period.

In order to label the shocks adequately based on Figure 2, we rely on former replication studies by Herwartz and Lütkepohl (2014) and Lanne and Luoto (2016). In the last column of Figure 2 we almost exactly replicate the responses to a monetary policy shock obtained by the method of Herwartz and Lütkepohl (2014). Also in line with the results of Uhlig (2005) and Lanne and Luoto (2016), the sign pattern suggested in Peersman (2005) is thereby not replicated. Furthermore,

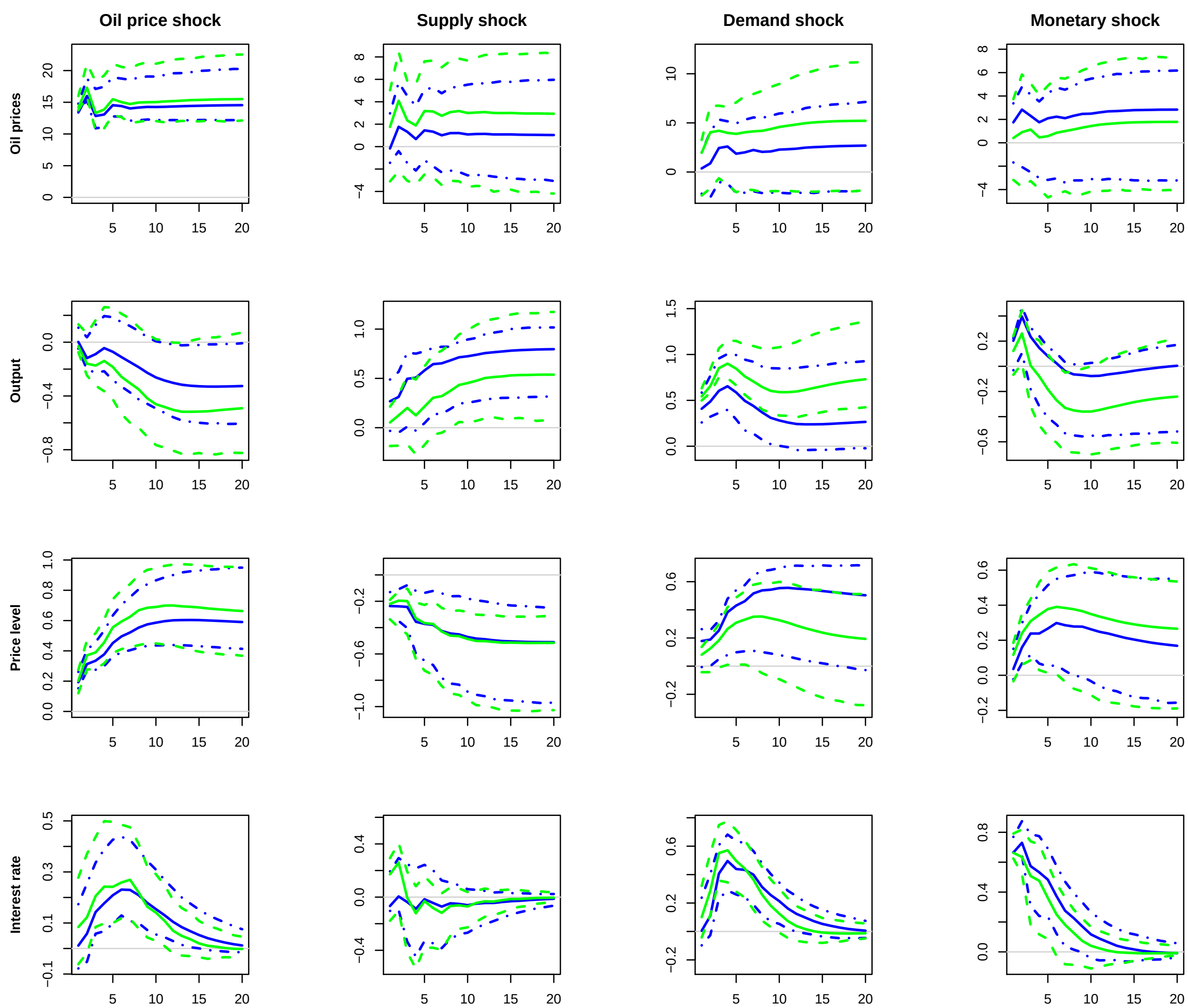

Figure 2: Impulse response functions based on identification by means of distance covariance for samples 1980Q1-2002Q2 (green, dashed confidence intervals) and 1980Q1-2007Q4 (blue, dotdashed confidence intervals).

Lanne and Luoto (2016) argue that only the oil price shock can be fully reproduced holding the suggested signs in the on-impact matrix. Acknowledging higher uncertainty in the instantaneous responses, we therefore label the first shock an oil price shock. The supply and demand shock both lead to insignificant responses in the associated variables and thus, might not be identifiable. However, the assigned labels appear economically reasonable and further support the results of Herwartz and Lütkepohl (2014) and Lanne and Luoto (2016). Overall, the impulse responses displayed in Figure 2 still indicate that a combination of shocks causes the slowdown in the short as well as in the long run. However, output does not seem to respond significantly to a monetary policy shock.

Decomposing output growth into the contribution of structural shocks in each time period provides further evidence on the causes of negative economic growth in 2001. Figure 3 shows the corresponding historical decompositions starting in 1995 up to 2007 (calculated as described in Lütkepohl, 2011). Based on Figure 3, the recession in 2001 is attributed to a combination of shocks which is in line with the conclusions drawn in Peersman (2005). Yet the size and direction of the contributions vary throughout the time periods of output declines. While in the third quarter of 2001 all shocks dampen output growth with roughly the same impact, their contributions in early 2001 differs. The aggregate demand shock provokes the largest negative contribution in quarter 1 of 2001 which is subsequently slightly positive in quarters 2 and 4. Throughout 2001 monetary policy further reduces output growth while the contribution becomes positive not before early 2002. Furthermore, the demand shock boosts output growth showing a positive contribution in early 2001 while the oil price shock contributes slightly negative in these periods. Overall, the historical decompositions show slight differences to the ones based on sign and traditional restrictions (results are displayed in Table I of Peersman, 2005). While the results appear reasonable, they still might be handled with care because of the weak validation of the non-normality assumption during the observed time period until 2007.

To avoid these sources of identification weaknesses and check robustness of the model, it might be worth to consider an extended sample until 2014Q2 including the period of the Great Recession. While this sample extension leads to non-Gaussianity of the structural shocks, we might argue that further variables are necessary to properly identify causes of economic slowdowns, in particular of the Great Recession. Furthermore, according to the replication study in Grant (2015) time varying parameter estimation might be better suited to derive at profound inferences on this extended time period. As an interesting aim for future research we leave these elaborated model modifications behind the scope of this paper.

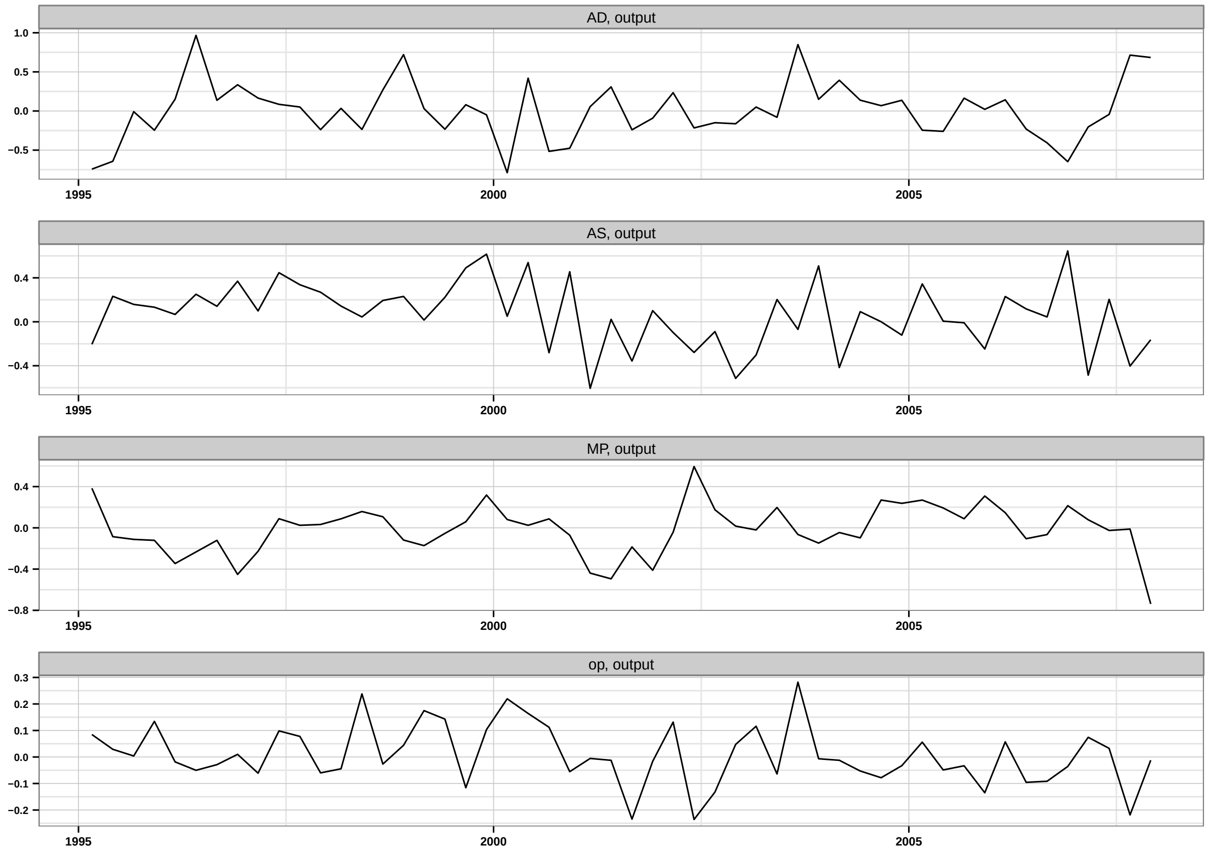

Figure 3: Historical decomposition of output growth attributed to the four shocks (oil price, aggregate supply, demand and monetary policy) based on independence based identification for sample 1995 to 2007.

4 Conclusions

Independence based identification by means of a nonparametric dependence measure allows for identification of a non-Gaussian SVAR model. We formulate identifiability in a more flexible way to overcome the limitations of this approach in the presence of multiple Gaussian structural shocks. In particular, besides identification of the whole system with at most one Gaussian component, the non-Gaussian shocks can be identified in systems which are closer to Gaussianity. Uniqueness of independence based identification of non-Gaussian structural shocks is proved theoretically. Extensions to higher dimensional systems are straightforward and might be of special interest if the analyst aims to derive economic conclusions about the response to specific shocks only (and these are non-Gaussian in their structural form).

Moreover, we retrieve these characteristics in a four dimensional macroeconomic VAR model. We revisit the study of Peersman (2005) to gain conclusive insights on macroeconomic causes of the early millennium slowdown over two different time horizons. We can uniquely identify two shocks, an oil price and a monetary policy shock, in the original sample until 2002. However, for inferences on the early millennium slowdown we advocate to consider the model ending in 2007Q4 because of non-Gaussianity of the structural shocks and a larger sample size compared to the original sample 1980Q1–2002Q2. Based on the extended sample, we obtain similar results as derived in the studies of Herwartz and Lütkepohl (2014) and Lanne and Luoto (2016). Furthermore, based on the historical decomposition of output growth into separate structural shocks we infer that a combination of shocks contributes to negative economic growth in 2001.

Appendix

Proof of Proposition 1. "$\Longleftarrow$" The proof of this implication is straightforward and, therefore, omitted.

"$\Longrightarrow$" We reformulate the $K \times K$ matrix C block wise by setting

$$C = \begin{pmatrix} C_1 & C_2 \\ C_3 & C_4 \end{pmatrix},$$

where, for instance, C_1 is a $k_1 \times k_1$ matrix. Consequently, the first k_1 Gaussian entries of ε_2 correspond to $\varepsilon_{2,1,\ldots,k_1} = \begin{pmatrix} C_1 & C_2 \end{pmatrix} \varepsilon_1$.

Suppose that one of the entries of the second block matrix C_2 would differ from zero. Following Lemma 9 of Comon (1994), the entry in ε_1 which is related to $\varepsilon_{2,1,\ldots,k_1}$ by this non zero entry in C_2 is Gaussian. This contradicts the assumption that the last $K - k_1$ components of ε_2 are non-Gaussian. Thus, $C_2 = 0_{k_1,K-k_1}$ and C_1 projects the first k_1 variables of ε_1 onto the first k_1 components of ε_2, i.e. $\varepsilon_{2,1,\ldots,k_1} = C_1 \varepsilon_{1,1,\ldots,k_1}$. Assuming that the components of ε_1 are independent and its first k_1 entries are normally distributed, matrix C_1 corresponds to an orthogonal matrix Q to preserve independence of the components in $\varepsilon_{2,1,\ldots,k_1} = Q\varepsilon_{1,1,\ldots,k_1}$ (see, for instance, Hyvärinen et al., 2001).

The matrix C is assumed to be orthogonal, i.e. $CC' = I_K$. Setting $C_2 = 0_{k_1,K-k_1}$ and $C_1 = Q$

the block wise formulation of this product corresponds to

$$
\begin{aligned}
CC' &= \begin{pmatrix} C_1 & C_2 \\ C_3 & C_4 \end{pmatrix} \begin{pmatrix} C_1' & C_3' \\ C_2' & C_4' \end{pmatrix} \\
&= \begin{pmatrix} C_1C_1' + C_2C_2' & C_1C_3' + C_2C_4' \\ C_3C_1' + C_4C_2' & C_3C_3' + C_4C_4' \end{pmatrix} = \begin{pmatrix} QQ' & QC_3' \\ C_3Q' & C_3C_3' + C_4C_4' \end{pmatrix}.
\end{aligned}
$$

Accordingly, all entries of the block matrices C_3Q' and QC_3' need to equal zero in order to obtain the identity matrix, $CC' = I_K$. As Q is orthogonal it has full rank. It follows $C_3Q' = 0_{K-k_1,k_1}$ and $QC_3' = 0_{k_1,K-k_1}$ if and only if $C_3 = 0_{K-k_1,k_1}$ with $0_{K-k_1,k_1}$ and $0_{K-k_1,k_1}$ corresponding to the $(K-k_1) \times k_1$ and $k_1 \times (K-k_1)$ zero matrices, respectively.

Hence, the product CC' can be written as

$$
CC' = \begin{pmatrix} QQ' & 0 \\ 0 & C_4C_4' \end{pmatrix}.
$$

Lastly, we consider the second part of ε_2 to determine the last block matrix C_4, i.e. $\varepsilon_{2,k_1+1,\dots,K} = \begin{pmatrix} 0 & C_4 \end{pmatrix} \varepsilon_1$. Matrix C_4 maps the non-Gaussian entries of ε_1 to the non-Gaussian entries of ε_2. Thus, this is an application of Comon's theorem: for independent components in ε_2, the matrix C_4 is the product of a diagonal and a permutation matrix ΛP following the derivations in Theorem 11 of Comon (1994). Finally,

$$
C = \begin{pmatrix} Q & 0 \\ 0 & \Lambda P \end{pmatrix} \quad \text{and} \quad CC' = \begin{pmatrix} QQ' & 0 \\ 0 & (\Lambda P)(\Lambda P)' \end{pmatrix} = \begin{pmatrix} I_{k_1} & 0 \\ 0 & I_{K-k_1} \end{pmatrix} = I_K.
$$

□

References

Blanchard, O. J., Quah, D., 1989. The dynamic effects of aggregate demand and supply disturbances. American Economic Review 79 (4), 655–73.

Boscolo, R., Pan, H., Roychowdhury, V. P., 2002. Beyond comon's identifiability theorem for independent component analysis. Vol. 2415 of Lecture Notes in Computer Science. Springer, pp. 1119–1124.

Chiu, C.-W. J., Mumtaz, H., Pinter, G., 2016. Var models with non-gaussian shocks. Centre for Macroeconomics (CFM) Dicussion Paper No 1609.

Comon, P., 1994. Independent component analysis, A new concept? Signal Processing 36 (3), 287–314.

Faust, J., 1998. The robustness of identified var conclusions about money. Carnegie-Rochester Conference Series on Public Policy 49 (1), 207–244.

Gavrilov, I., Pusev, R., 2015. normtest: Tests for the composite hypothesis of normality. R package version 1.1.

Gouriéroux, C., Monfort, A., Renne, J.-P., 2017. Statistical inference for independent component analysis: Application to structural var models. Journal of Econometrics 196, 111–126.

Grant, A. L., 2015. The early millennium slowdown: Replicating the peersman (2005) results. Journal of Applied Econometrics 32 (1), 224–232.

Herwartz, H., Lütkepohl, H., 2014. Structural vector autoregressions with Markov switching: combining conventional with statistical identification of shocks. J. Econometrics 183 (1), 104–116.

Herwartz, H., Plödt, M., 2016. The macroeconomic effects of oil price shocks: Evidence from a statistical identification approach. Journal of International Money and Finance 61, 30–44.

Hyvärinen, A., Karhunen, J., Oja, E., 2001. Independent Component Analysis. John Wiley & Sons.

Kilian, L., Lütkepohl, H., 2017. Structural Vector Autoregressive Analysis. Cambridge University Press, forthcoming.

Lanne, M., Luoto, J., 2016. Data-driven inference on sign restrictions in bayesian structural vector autoregression. CREATES Research Paper 2016-4.

Lanne, M., Meitz, M., Saikkonen, P., 2017. Identification and estimation of non-Gaussian structural vector autoregressions. J. Econometrics 196 (2), 288–304.

Lütkepohl, H., 2011. Vector autoregressive models. Eui working paper eco 2011/30, European University Institute, Florence.

Matteson, D. S., Tsay, R. S., 2017. Independent component analysis via distance covariance. Journal of the American Statistical Association 112, 623–637.

Moneta, A., Entner, D., Hoyer, P. O., Coad, A., 2013. Causal inference by independent component analysis: Theory and applications. Oxford Bulletin of Economics and Statistics 75 (5), 705–730.

Nordhausen, K., Oja, H., Tyler, D. E., Virta, J., 2016. ICtest: Estimating and Testing the Number of Interesting Components in Linear Dimension Reduction. R package version 0.2.

Peersman, G., 2005. What caused the early millennium slowdown? Evidence based on vector autoregressions. J. Appl. Econometrics 20 (2), 185–207.

Risk, B. B., James, N. A., Matteson, D. S., 2015. steadyICA: ICA and Tests of Independence via Multivariate Distance Covariance. R package version 1.0.

Sims, C. A., 1980. Macroeconomics and reality. Econometrica 48 (1), 1–48.

Székely, G. J., Rizzo, M. L., Bakirov, N. K., 2007. Measuring and testing dependence by correlation of distances. Ann. Statist. 35 (6), 2769–2794.

Uhlig, H., 2005. What are the effects of monetary policy on output? results from an agnostic identification procedure. Journal of Monetary Economics 52 (2), 381–419.