Allen-Cahn equation in periodic media

Exercises 2

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Plane-like minimizers in periodic media - Exercise sheet*

Let $a_{ij} : \mathbb{R}^n \to \mathbb{R}$ be, for any $i, j = 1, \ldots, n$, be a Lipschitz function, periodic under integer translations, i.e. $a_{ij}(x + v) = a_{ij}(x)$, for any $v \in \mathbb{Z}^n$. Assume moreover that the a_{ij}'s form a symmetric, bounded and uniformly elliptic matrix, that is

$$a_{ij} = a_{ji} \qquad \text{and} \qquad \lambda|\xi|^2 \leq a_{ij}\xi_i\xi_j \leq \Lambda|\xi|^2, \, \forall\, \xi \in \mathbb{R}^n,$$

for some $0 < \lambda \leq \Lambda < +\infty$. Let $W : \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}$ be a *double well* potential, i.e. a non-negative function $(x, u) \mapsto W(x, u)$ measurable and $\mathbb{Z}^n$-periodic in the x variable, C^2 in the u variable, such that

$$\begin{aligned} W(x, \pm 1) = W_u(x, \pm 1) = 0, \quad W_{uu}(x, \pm 1) &> 0, \qquad \text{for any } x \in \mathbb{R}^n, \\ W(x, u) &> 0, \qquad \text{for any } x \in \mathbb{R}^n, u \in (-1, 1). \end{aligned}$$

Given any $\omega \in \mathbb{Q}^n \setminus \{0\}$, consider on $\mathbb{R}^n$ the equivalence relation $\sim$ defined by

$$x \sim y \qquad \text{if and only if} \qquad y - x = k \in \mathbb{Z}^n, \text{ with } \omega \cdot k = 0.$$

Set $\widetilde{\mathbb{R}}^n := \mathbb{R}^n/\sim$. Furthermore, we consider the strip

$$S^\omega_M := \{x \in \mathbb{R}^n : \omega \cdot x \in [0, M]\},$$

and the related quotient $\widetilde{S}^\omega_M := S^\omega_M/\sim$. In the following we will freely identify a quotient with any of its corresponding fundamental domains of $\mathbb{R}^n$. For any measurable $\Omega \subset \mathbb{R}^n$, we introduce the energy functional

$$\mathcal{G}_\Omega(u) := \int_\Omega a_{ij}(x)u_i(x)u_j(x) + W(x, u(x))\, dx, \qquad \text{for } u \in H^1_{\text{loc}}(\Omega),$$

considering it set to $+\infty$ whenever the integral does not converge.

Exercise 1.
Prove that cutting off the values above 1 and below −1 from a function u does not increase its energy $\mathcal{G}_\Omega$, that is

$$\mathcal{G}_\Omega(\min\{u, 1\}), \mathcal{G}_\Omega(\max\{u, -1\}) \leq \mathcal{G}_\Omega(u).$$

Due to this fact we will always implicitly assume the functions to have values in $[-1, 1]$. Introduce now the class of *admissible functions* on $\widetilde{\mathbb{R}}^n$

$$\mathcal{X}^\omega_M := \{u \in H^1_{\text{loc}}(\widetilde{\mathbb{R}}^n) : u(x) \geq 9/10 \text{ if } \omega \cdot x \leq 0, \text{ and } u(x) \leq -9/10 \text{ if } \omega \cdot x \geq M\},$$

and let $\mathcal{M}^\omega_M$ denote the set of the absolute minimizers of $\mathcal{G}_{\widetilde{\mathbb{R}}^n}$ in $\mathcal{X}^\omega_M$, i.e.

$$\mathcal{M}^\omega_M := \{u \in \mathcal{X}^\omega_M : \mathcal{G}_{\widetilde{\mathbb{R}}^n}(u) = \min\{\mathcal{G}_{\widetilde{\mathbb{R}}^n}(v) : v \in \mathcal{X}^\omega_M\}\}.$$

Exercise 2.
Prove the following two assertions in order to obtain the existence of a global minimizer of $\mathcal{G}_{\widetilde{\mathbb{R}}^n}$ in $\mathcal{X}^\omega_M$ periodic with respect to $\sim$.

(i) Show that the functional $\mathcal{G}_{\widetilde{\mathbb{R}}^n}$ is not identically infinite on $\mathcal{X}^\omega_M$ by giving the explicit example of a function $u_0 \in \mathcal{X}^\omega_M$ with $\mathcal{G}_{\widetilde{\mathbb{R}}^n}(u_0) < +\infty$;
(Hint: consider the piecewise linear function $u_0 : \widetilde{\mathbb{R}}^n \to \mathbb{R}$ having constant values 1 for $\omega \cdot x \leq 0$ and −1 for $\omega \cdot x \geq M$.)

*The exercises listed here will be carried out as part of the tutoring activities on August, 28th.

(ii) *Show then that $\mathcal{M}^\omega_M$ is not empty;*
(Hint: use the standard direct method of calculus of variations. It could be useful to introduce the set $\widetilde{D}_R := \{x \in \widetilde{\mathbb{R}}^n : \omega \cdot x \in [-R, R]\ \}$, with $R > M$, to cope with some difficulties due to the unboundedness of $\widetilde{\mathbb{R}}^n$.)

It can be shown - using techniques *à la* De Giorgi or applying the regularity theory for elliptic PDEs to the Euler-Lagrange equation of $\mathcal{G}$ - that there exists $\alpha \in (0,1)$ such that, given any compact $K \subset \widetilde{\mathbb{R}}^n$,

$$\|u\|_{C^{0,\alpha}(K)} \leq C, \qquad \text{for any } u \in \mathcal{M}^\omega_M,$$

for some constant $C > 0$, so that, in particular, $\mathcal{M}^\omega_M \subset C^{0,\alpha}_{\text{loc}}(\widetilde{\mathbb{R}}^n)$. Now we introduce the periodic function

$$u^\omega_M(x) := \inf_{u \in \mathcal{M}^\omega_M} u(x), \qquad \text{for any } x \in \widetilde{\mathbb{R}}^n,$$

which will be referred to as the *minimal minimizer* of $\mathcal{X}^\omega_M$.

Exercise 3.
Prove that $u^\omega_M \in C^{0,\alpha}_{\text{loc}}(\widetilde{\mathbb{R}}^n)$.

Exercise 4.
Given $\alpha, \beta \geq 0$, consider the strip

$$S^\omega_{-\alpha, M-\beta} := \{x \in \mathbb{R}^n : \omega \cdot x \in [-\alpha, M - \beta]\},$$

and the associated space of minimizers $\mathcal{M}_{-\alpha, M-\beta}$. Prove the following statements.

(i) *If $u, v \in H^1_{\text{loc}}(\widetilde{\mathbb{R}}^n)$, then*

$$\mathcal{G}_{\widetilde{\mathbb{R}}^n}(\min\{u,v\}) + \mathcal{G}_{\widetilde{\mathbb{R}}^n}(\max\{u,v\}) \leq \mathcal{G}_{\widetilde{\mathbb{R}}^n}(u) + \mathcal{G}_{\widetilde{\mathbb{R}}^n}(v);$$

(ii) *If $u \in \mathcal{M}^\omega_M$ and $v \in \mathcal{M}_{-\alpha, M-\beta}$, then $\min\{u,v\} \in \mathcal{M}_{-\alpha, M-\beta}$ and $\max\{u,v\} \in \mathcal{M}^\omega_M$. In particular, $(\mathcal{M}^\omega_M, \min, \max)$ is a lattice;*

(iii) *$u^\omega_M \in \mathcal{M}^\omega_M$.*
(Hint: use Arzelà-Ascoli theorem.)

For $k \in \mathbb{R}^n$, denote with $\mathcal{T}_k$ the translation of vector k, both for sets and functions:

$$\mathcal{T}_k E := \{x + k : x \in E\}, \qquad \mathcal{T}_k f := f(\cdot - k).$$

Given a set $E \subset \mathbb{R}^n$ and a vector $\varpi \in \mathbb{R}^n$, we say that E satisfies the *Birkhoff property* with respect to ϖ (or are ϖ-*Birkhoff*) if

- For any $k \in \mathbb{Z}^n$ such that $\varpi \cdot k \leq 0$, we have $\mathcal{T}_k E \subset E$;
- For any $k \in \mathbb{Z}^n$ such that $\varpi \cdot k \geq 0$, we have $\mathcal{T}_k E \supset E$.

Exercise 5.
Check the following geometric properties.

(i) *If $\{E_\alpha\}_{\alpha \in A}$ is a family of ϖ-Birkhoff sets, then both $\cap_{\alpha \in A} E_\alpha$ and $\cup_{\alpha \in A} E_\alpha$ are ϖ-Birkhoff;*

(ii) *If E is Birkhoff with respect to ϖ, then E^c is Birkhoff with respect to $-\varpi$;*

(iii) There exists a constant $\rho > 0$ depending on n such that if a ϖ-Birkhoff set E contains a ball of radius ρ, then it contains also a strip of width 1 with sides orthogonal to ϖ which itself contains the center of the ball.

Exercise 6.
Given $\theta \in \mathbb{R}$, show that the superlevel set $\{u^\omega_M > \theta\}$ is ω-Birkhoff.
(Hint: Use point (ii) of Exercise 4 with u^ω_M and $\mathcal{T}_k u^\omega_M$.)

Fix now a fundamental domain for $\sim$ of the form $F = Q_F \times \mathbb{R}$, with Q_F a $(n-1)$-dimensional hyperrectangle in any hyperplane orthogonal to ω. Introduce the *doubled* relation $\sim_D$ defined by setting

$$x \sim_D y \qquad \text{if and only if} \qquad y - x = k \in (2\mathbb{Z})^n, \text{ with } \omega \cdot k = 0,$$

and choose $D = Q_D \times \mathbb{R}$ to be a fundamental domain for $\sim_D$ with $Q_F \subset Q_D$. We also introduce the space of admissible functions $\mathcal{X}^\omega_{D,M}$ and that of minimizers $\mathcal{M}^\omega_{D,M}$ relative to $D = \mathbb{R}^n / \sim_D$. Clearly, $\mathcal{X}^\omega_M \subset \mathcal{X}^\omega_{D,M}$ and $\mathcal{G}_D(u) = 2^{n-1}\mathcal{G}_{\widetilde{\mathbb{R}}^n}(u)$, for any $u \in \mathcal{X}^\omega_M$.
In the following exercise we show that the minimal minimizer enjoys the so-called *doubling property* or *no-symmetry-breaking property*, that is the minimal minimizer is still a minimizer with respect to functions of doubled periodicity.

Exercise 7. *Let $u^\omega_{D,M}$ denotes the minimal minimizer of $\mathcal{X}^\omega_{D,M}$. Show that*

$$\{\mathcal{T}_k u^\omega_{D,M} = \ell\} = \{u^\omega_{D,M} = \ell\}, \qquad \textit{for any } \ell \in \mathbb{R},\ k \in \mathbb{Z}^n \textit{ with } \omega \cdot k = 0,$$

deduce that $u^\omega_{D,M}$ is periodic with respect to $\sim$ and thus that it defines an element in $\mathcal{X}^\omega_M$. Use this to conclude that u^ω_M and $u^\omega_{D,M}$ coincide.