Introduction to Spectral Theory

Third lecture: Some applications

Ingo Witt (Göttingen)

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Stone's formula

Theorem

Let $A \in \mathcal{L}(\mathcal{H})$ be self-adjoint. Then, for any $a < b$,

$$\frac{1}{2}\left(E_{[a,b]} + E_{(a,b)}\right) = s\text{-}\lim_{\epsilon \to +0} \frac{1}{2\pi \mathrm{i}} \int_a^b \left(R(\lambda + \mathrm{i}\epsilon, A) - R(\lambda - \mathrm{i}\epsilon, A)\right) \mathrm{d}\lambda.$$

Proof Compute

$$f_\epsilon(\mu) = \frac{1}{2\pi \mathrm{i}} \int_a^b \left(\frac{1}{\mu - \lambda - \mathrm{i}\epsilon} - \frac{1}{\mu - \lambda + \mathrm{i}\epsilon} \right) \mathrm{d}\lambda \xrightarrow{\epsilon \to +0} \begin{cases} 0, & \mu \notin [a,b], \\ 1/2, & \mu = a \text{ or } \mu = b, \\ 1, & \mu \in (a,b). \end{cases}$$

In view of $\sup_{0<\epsilon\le 1} \|f_\epsilon\|_\infty < \infty$, the result follows by invoking the functional calculus. $\square$

The lattice Laplacian

Take the Hilbert space $\mathcal{H}$ to be $\ell^2(\mathbb{Z}^d)$ equipped with the counting measure. As A we take the operator

$$(A\psi)(n) = \sum_{|m-n|=1} \psi(m), \quad n \in \mathbb{Z}^d.$$

Note that the discrete Laplacian is $-2d + A$ (approximation of the continuous Laplacian by second differences); we have chosen to ignore the shift by $2d$.

Proposition

The spectrum of A equals $[-2d, 2d]$, and it is purely absolutely continuous.

Sketch of proof

Let $F\colon L^2([0,2\pi]^d) \to \ell^2(\mathbb{Z}^d)$ be the (periodic) Fourier transform, i.e.,

$$(Ff)(n) = (2\pi)^{-d} \int_{[0,2\pi]^d} \mathrm{e}^{-\mathrm{i}x\cdot n} f(x)\,\mathrm{d}x, \quad n \in \mathbb{Z}^d.$$

F^* provides a spectral resolution of A. Indeed, a direct calculation reveals that

$$F^*AF \text{ is multiplication by } 2\sum_{j=1}^{d} \cos x_j,$$

and the result follows. □

The Schrödinger equation

The initial-value problem for the Schrödinger equation is

$$i\partial_t u = Au, \quad u(0) = \varphi,$$

where $A\colon \mathscr{D}(A) \subseteq \mathcal{H} \to \mathcal{H}$ is a self-adjoint operator on $\mathcal{H}$ and $\varphi \in \mathscr{D}(A)$. Its unique solution $u \in \mathscr{C}(\mathbb{R}; \mathscr{D}(A)) \cap \mathscr{C}^1(\mathbb{R}; \mathcal{H})$ is given by

$$u(t) = e^{-itA}\varphi, \quad t \in \mathbb{R}.$$

By Stone's theorem, $\{e^{-itA}\}_{t\in\mathbb{R}}$ is a strongly continuous unitary group on $\mathcal{H}$ (norm continuous if $A \in \mathcal{L}(\mathcal{H})$).

Remark The solution u to the inhomogeneous problem

$$i\partial_t u = Au + f(t), \quad u(0) = \varphi,$$

is given by

$$u(t) = e^{-itA}\varphi - i\int_0^t e^{-i(t-s)A} f(s)\, ds.$$

Long-term behavior

Basic fact Spectral properties of A determine dynamical properties of the Schrödinger evolution as $t \to \pm\infty$.

- (Bound states) Let $A\varphi = \lambda\varphi$. Then $\mathrm{e}^{-\mathrm{i}tA}\varphi = \mathrm{e}^{-\mathrm{i}t\lambda}\varphi$ and

$$\langle B\mathrm{e}^{-\mathrm{i}tA}\varphi, \mathrm{e}^{-\mathrm{i}tA}\varphi\rangle \text{ is independent of } t.$$

  Here, B is a self-adjoint operator that is relatively bounded with respect to A (i.e., $\mathscr{D}(B) \supseteq \mathscr{D}(A)$).

- (RAGE theorem, after Ruelle, Amrein, Georgescu, and Enss) Let $\psi \in \mathcal{H}_{\mathrm{ac}}(A) \oplus \mathcal{H}_{\mathrm{sc}}(A)$ and $B \in \mathcal{L}(\mathcal{H})$ be relatively compact with respect to A. Then

$$\frac{1}{2T}\int_{-T}^{T} \|B\mathrm{e}^{-\mathrm{i}tA}\psi\|^2 \, dt \to 0 \text{ as } T \to \infty.$$

Fredholm operators

Definition

$T \in \mathcal{L}(\mathcal{H})$ is said to be Fredholm if $\dim \ker T < \infty$ and $\dim \operatorname{coker} T < \infty$ (recall that $\operatorname{coker} T = \mathcal{H}/\operatorname{ran} T$). In this case, $\operatorname{ran} T \subseteq \mathcal{H}$ is closed and the integer

$$\operatorname{ind} T = \dim \ker T - \dim \operatorname{coker} T$$

is called the index of T.

We write $\mathcal{F}(\mathcal{H}, \mathcal{H}')$ for the space of Fredholm operators and $\mathcal{F}(\mathcal{H})$ in case $\mathcal{H} = \mathcal{H}'$.

Remarks

(a) Let $\mathcal{H}$, $\mathcal{H}'$ be finite-dimensional. Then, for any linear operator $T \colon \mathcal{H} \to \mathcal{H}'$,

$$\operatorname{ind} T = \dim \mathcal{H} - \dim \mathcal{H}'.$$

(b) Every self-adjoint Fredholm operator has index zero.

Theorem

(a) *If* $T \in \mathcal{F}(\mathcal{H}, \mathcal{H}')$, $S \in \mathcal{F}(\mathcal{H}', \mathcal{H}'')$, *then* $ST \in \mathcal{F}(\mathcal{H}, \mathcal{H}'')$ *and*

$$\operatorname{ind}(ST) = \operatorname{ind}(T) + \operatorname{ind}(S).$$

(b) *Let* $T \in \mathcal{F}(\mathcal{H}, \mathcal{H}')$ *and* $K \in \mathcal{K}(\mathcal{H}, \mathcal{H}')$. *Then* $T + K \in \mathcal{F}(\mathcal{H}, \mathcal{H}')$ *and*

$$\operatorname{ind}(T + K) = \operatorname{ind} T.$$

(c) $\mathcal{F}(\mathcal{H}, \mathcal{H}') \subseteq \mathcal{L}(\mathcal{H}, \mathcal{H}')$ *is norm open and the map*

$$\operatorname{ind}\colon \mathcal{F}(\mathcal{H}, \mathcal{H}') \to \mathbb{Z}$$

is continuous (*i.e., constant on connected components*).

Fredholm operators

Further properties

Theorem

$T \in \mathcal{L}(\mathcal{H}, \mathcal{H}')$ is a Fredholm operator if and only if there exists an operator $S \in \mathcal{L}(\mathcal{H}', \mathcal{H})$ such that

$$ST - \mathrm{I}_{\mathcal{H}} \in \mathcal{K}(\mathcal{H}), \quad TS - \mathrm{I}_{\mathcal{H}'} \in \mathcal{K}(\mathcal{H}').$$

S is then called a (Fredholm) parametrix to T. Note that S is a Fredholm operator as well, and $\operatorname{ind} S = -\operatorname{ind} T$.

Typical example Elliptic differential or pseudo-differential operators on closed manifolds acting between Sobolev spaces.

Spectral flow

Heuristics

Let $\dim \mathcal{H} = \infty$. Define

$$\mathcal{F}_{\mathrm{sa}}(\mathcal{H}) = \{T \in \mathcal{F}(\mathcal{H}) \mid T \text{ is self-adjoint}\}.$$

For $T \in \mathcal{F}_{\mathrm{sa}}(\mathcal{H})$, there exists an $\epsilon > 0$ such that $\sigma_e(T) \cap (-\epsilon, \epsilon) = \emptyset$. Given a norm continuous path $\gamma\colon [0, 1] \to \mathcal{F}_{\mathrm{sa}}(\mathcal{H})$, this property allows to define the spectral flow $\mathrm{sf}(\gamma)$ as the net number of eigenvalues (counted with multiplicities) which pass through zero in the positive direction as t goes from 0 to 1.

Lemma

Let $T \in \mathcal{F}_{\mathrm{sa}}(\mathcal{H})$. Then there exist a neighborhood $\mathcal{N}$ of T in $\mathcal{F}_{\mathrm{sa}}(\mathcal{H})$ and an $a > 0$ so that $S \mapsto \chi_{[-a,a]}(S)$ is a norm continuous, finite-rank projection-valued function on $\mathcal{N}$.

Spectral flow

Definition

Now given a norm continuous path $\gamma\colon [0,1] \to \mathcal{F}_{\mathrm{sa}}(\mathcal{H})$, one finds a partition $0 = t_0 < t_1 < \ldots < t_{k-1} < t_k = 1$ and numbers $a_j > 0$ for $j = 1, \ldots, k$ so that the function $t \mapsto \chi_{[-a_j,a_j]}(\gamma(t))$ is norm continuous and of finite rank on the interval $[t_{j-1}, t_j]$.

Definition

We set

$$\mathrm{sf}(\gamma) = \sum_{j=1}^{k} \Big(\mathrm{rank}\big(\chi_{[0,a_j]}(\gamma(t_j))\big) - \mathrm{rank}\big(\chi_{[0,a_j]}(\gamma(t_{j-1}))\big)\Big).$$

Lemma

The integer $\mathrm{sf}(\gamma)$ *is well-defined, i.e., it is independent of all the choices involved in its definition (including the parametrization of* γ *when keeping the orientation).*

Spectral flow

Properties

Theorem

Spectral flow has the following properties:

(a) (Homotopy invariance) $\mathrm{sf}(\gamma_0) = \mathrm{sf}(\gamma_1)$ *if* γ_0, γ_1 *are homotopic in* $\mathcal{F}_{\mathrm{sa}}(\mathcal{H})$ *with fixed endpoints.*

(b) (Concatenation) $\mathrm{sf}(\gamma_1 \circ \gamma_0) = \mathrm{sf}(\gamma_0) + \mathrm{sf}(\gamma_1)$ *if* $\gamma_0(1) = \gamma_1(0)$.

(c) (Unitary invariance) $\mathrm{sf}(U\gamma U^*) = \mathrm{sf}(\gamma)$ *for each unitary operator* U.

(d) (Additivity) $\mathrm{sf}(\gamma_0 \oplus \gamma_1) = \mathrm{sf}(\gamma_0) + \mathrm{sf}(\gamma_1)$.

(e) (Triviality) $\mathrm{sf}(\gamma) = 0$ *if* $0 \notin \sigma(\gamma(t))$ *for all* $t \in [0, 1]$.

Components of $\mathcal{F}_{\text{sa}}(\mathcal{H})$

$\mathcal{F}_{\text{sa}}(\mathcal{H})$ has three components determined by the essential spectrum:

$$\mathcal{F}^{\pm}_{\text{sa}}(\mathcal{H}) = \{T \in \mathcal{F}_{\text{sa}}(\mathcal{H}) \mid \sigma_{\text{e}}(T) \subset \mathbb{R}_{\pm}\},$$
$$\mathcal{F}^{0}_{\text{sa}}(\mathcal{H}) = \{T \in \mathcal{F}_{\text{sa}}(\mathcal{H}) \mid \sigma_{\text{e}}(T) \cap \mathbb{R}_{\pm} \neq \emptyset\}.$$

$\mathcal{F}^{\pm}_{\text{sa}}(\mathcal{H})$ are contractible.

Theorem

The map

$$\text{sf}\colon \pi_1(\mathcal{F}^{0}_{\text{sa}}(\mathcal{H})) \to \mathbb{Z}$$

induced by spectral flow is an isomorphism.