On the Distribution of the Adaptive LASSO Estimator (pt I)

Ulrike Schneider
(joint with B.M. Pötscher)

University of Vienna

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Outline

Penalized LS (ML) Estimators

Linear regression model

$$\mathbf{y} = \theta_1 \mathbf{x}_{.1} + \ldots \theta_k \mathbf{x}_{.k} + \boldsymbol{\varepsilon}$$

- response $\mathbf{y} \in \mathbb{R}^n$ (known)
- regressors $\mathbf{x}_{.i} \in \mathbb{R}^n$, $1 \leq i \leq k$ (known)
- errors $\boldsymbol{\varepsilon} \in \mathbb{R}^n$ (unknown)
- parameter vector $\boldsymbol{\theta} = (\theta_1, \ldots, \theta_k)' \in \mathbb{R}^k$ (unknown)

A penalized least-squares (LS) estimator $\hat{\theta}$ for $\boldsymbol{\theta}$ is given by

$$\hat{\theta} = \arg\min_{\boldsymbol{\theta} \in \mathbb{R}^k} \underbrace{\|\mathbf{y} - X\boldsymbol{\theta}\|^2}_{\text{likelihood or LS -part}} + \underbrace{\lambda_n \, p(\boldsymbol{\theta})}_{\text{penalty}}$$

$\lambda_n > 0$ is a tuning parameter ($\lambda_n = 0$ corresponds to unpenalized/ ordinary LS), $X = [\mathbf{x}_{.1}, \ldots, \mathbf{x}_{.k}]$ the $n \times k$ regression matrix.

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Penalized LS (ML) Estimators (cont'd)

Clearly, different penalties give rise to different estimators.

- General class of Bridge-estimators (Frank & Friedman, 1993) using l_γ - type penalties

$$\lambda_n p(\boldsymbol{\theta}) = \lambda_n \sum_{i=1}^{k} |\theta_i|^\gamma$$

$\gamma = 2$: Ridge-estimator (Hoerl & Kennard, 1970)
$\gamma = 1$: LASSO (Tibshirani, 1996).

- Hard- and soft-thresholding estimators.
- Smoothly clipped absolute deviation (SCAD) estimator (Fan & Li, 2001).
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Relationship to classical PMS-estimators

Brigde-estimators satisfy

$$\min_{\boldsymbol{\theta}\in\mathbb{R}^k} \|y - X\boldsymbol{\theta}\|^2 + \lambda_n \sum_{i=1}^{k} |\theta_i|^\gamma \quad (0 < \gamma < \infty)$$

For $\gamma \to 0$, get

$$\min_{\boldsymbol{\theta}\in\mathbb{R}^k} \|y - X\boldsymbol{\theta}\|^2 + \lambda_n \,\mathrm{card}\{i : \theta_i \neq 0\}$$

which yields a minimum C_p-type procedure such as AIC and BIC.
(l_γ-type penalty with "$\gamma = 0$")

Relationship to classical PMS-estimators (cont'd)

- For "$\gamma = 0$" procedures are computationally expensive.
- For $\gamma > 0$ (Bridge) estimators are more computationally tractable, especially for $\gamma \geq 1$ (convex objective function).
- For $\gamma \leq 1$, estimators perform model selection

$$P_{n,\theta}(\hat{\theta}_i = 0) > 0 \quad \text{if } \theta_i = 0.$$

  Same for SCAD, hard- and soft-thresholding. Phenomenon is more pronounced for smaller γ.
- $\gamma = 1$ (LASSO and adaptive LASSO) as compromise between the wish to detect zeros and computational simplicity. (SCAD leads to a non-convex optimization problem.)

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The PLS estimator(s) we treat in the following can be viewed to simultaneously perform model selection and parameter estimation.

Some terminology (model selection)

- Consistent model selection – Zero coefficients are found with asymptotic probability equal to 1.

$$\lim_{n\to\infty} P_{n,\theta}(\hat{\theta}_i = 0) = 1 \quad \text{whenever } \theta_i = 0 \quad (1 \leq i \leq k)$$
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An estimator performing consistent model selection is said to have the sparsity property.

- Conservative model selection – Zero coefficients are found with asymptotic probability less than 1.

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- Consistent vs. conservative model selection can in our context be driven by the asymptotic behavior of the tuning parameters λ_n. Also called "sparsely" vs. "non-sparsely" tuned procedures.

- Oracle property – Asymptotic distribution coincides with the one of the infeasible unpenalized estimator using the true zero restrictions (with VC-matrix Σ_{θ}).

$$n^{1/2}(\hat{\boldsymbol{\theta}} - \boldsymbol{\theta}) \rightarrow N(\mathbf{0}, \Sigma_{\boldsymbol{\theta}})$$

  Seems to suggest that $\hat{\boldsymbol{\theta}}$ performs as well as if we would know the true zero coefficients of $\boldsymbol{\theta}$.

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Literature on distributional properties of PLSEs

- Knight & Fu, 2000. Moving-parameter asymptotics for non-sparsely tuned LASSO and Bridge estimators in general.
- Fan & Li, 2001. Fixed-parameter asymptotics for SCAD.
- Zou, 2006. Fixed-parameter asymptotics for sparsely-tuned LASSO and adaptive LASSO.
- Additional papers establishing the oracle property for sparsely-tuned PLSEs and related estimators within a fixed-parameter framework.

  Fan & Li (2002, 2004), Bunea (2004), Bunea & McKeague (2005), Wang & Leng (2007), Li & Liang (2007), Wang, G. Li, & Tsai (2007), Zhang & Li (2007), Wang, R. Li, & Tsai (2007), Zou & Yuan (2008), Zou & Li (2008), Johnson, Lin, & Zeng (2008), . . .

Literature on distributional properties of PLSEs (cont'd)

This talk is based on

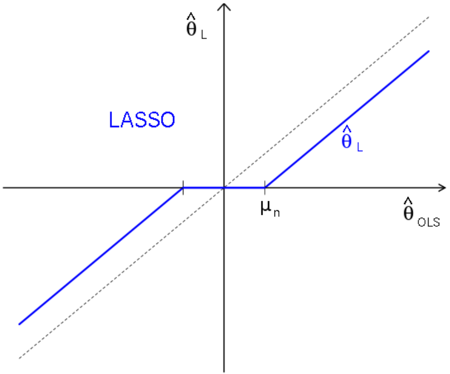
- Pötscher & Leeb, 2007. Finite-sample distribution, moving-parameter asymptotics for hard-thresholding, LASSO, and SCAD. Impossibility result for the estimation of the cdf.
- Pötscher & Schneider, 2007. Analogous results for the adaptive LASSO.
- Pötscher & Schneider, 2008. Finite-sample and asymptotic coverage probabilities of confidence sets for hard-thresholing, LASSO, ad. LASSO.

Definition of the (adaptive) LASSO estimator $\hat{\theta}_{AL}$

LASSO estimator (Tibshirani, 1996)

$$\hat{\boldsymbol{\theta}}_{L} = \underset{\boldsymbol{\theta} \in \mathbb{R}^k}{\arg\min} \ \|\mathbf{y} - X\boldsymbol{\theta}\|^2 + 2n\mu_n \sum_{i=1}^{k} |\theta_i| \qquad \mu_n > 0$$

Tuning parameter $\lambda_n = 2n\mu_n$. For $k = 1$:



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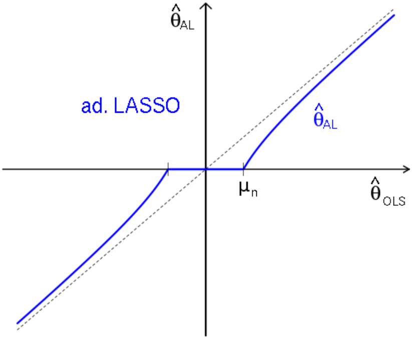

Two regimes for consistency

In terms of model selection consistency, two possible regimes for the tuning parameter μ_n arise.

1. The case $\mu_n \to 0$ and $n^{1/2}\mu_n \to m$, $0 \leq m < \infty$, corresponds to conservative model selection (non-sparsely tuned).
2. The case $\mu_n \to 0$ and $n^{1/2}\mu_n \to \infty$ corresponds to consistent model selection (sparsely tuned).

Remark (estimation consistency).
If $\mu_n \not\to 0$, then $\hat{\theta}_{\text{AL}}$ is not even consistent for θ. Therefore, $\mu_n \to 0$ is a "basic condition".

We will focus on 2 here, also discuss 1 .

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Asymptotics in the consistent case

Zou (2006) "oracle property"

Suppose $X'X/n \to Q > 0$ and $\varepsilon_t \overset{\text{iid}}{\sim} (0, \sigma^2)$.
If $\mu_n \to 0$ and $n^{1/2}\mu_n \to \infty$ and additionally $n^{1/4}\mu_n \to 0$, then

$$n^{1/2}(\hat{\boldsymbol{\theta}}_{\text{AL}} - \boldsymbol{\theta}) \to N(\mathbf{0}, \Sigma_\theta),$$

where Σ_θ is the asymptotic VC-matrix of the restricted LS-estimator based on the unknown true zero restrictions.

Questions

- Does this theorem provide meaningful insights? Finite-sample distribution?
- Asymptotic behavior under regime (1) ?
- What if condition $n^{1/4}\mu_n \to 0$ is dropped in (2) ?
- Pointwise vs. uniform consistency rates?
- Properties of confidence intervals?
- Estimability of finite-sample distribution?

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We answer these questions within a normal linear regression model and address the non-orthogonal case in a simulation study.

Explicit solution in a simple model

- X is non-stochastic $(n \times k)$, $rk(X) = k$.
- $\varepsilon \sim N_n(0, \sigma^2 \mathcal{I}_n)$
- For the theoretical analysis, assume that σ^2 is known and that $X'X$ is diagonal, in particular $X'X = n\mathcal{I}_k$.
- Remove these assumptions for simulation results concerning the finite-sample distribution.

Wlog consider Gaussian location model $y_1, \dots, y_n \overset{\text{iid}}{\sim} N(\theta, 1)$.
Then $\hat{\theta}_{\text{OLS}} = \bar{y}$ with $\hat{\theta}_{\text{OLS}} \sim N(\theta, 1/n)$ and

$$\hat{\theta}_{\text{AL}} = \begin{cases} 0 & \text{if } |\bar{y}| \leq \mu_n \\ \bar{y} - \mu_n^2/\bar{y} & \text{if } |\bar{y}| > \mu_n \end{cases}$$

Selects between restricted $\{N(0,1)\}$ and full model $\{N(\theta, 1) : \theta \in \mathbb{R}\}$

The finite-sample distribution of $\hat{\theta}_{\text{AL}}$

The cdf $F_{n,\theta}(x) = P_{n,\theta}(n^{1/2}(\hat{\theta}_{\text{AL}} - \theta) \leq x)$ of $\hat{\theta}_{\text{AL}}$ is given by

$$\mathbf{1}(n^{1/2}\theta + x \geq 0)\, \Phi\left(z^{(2)}_{n,\theta}(x)\right) + \mathbf{1}(n^{1/2}\theta + x < 0)\, \Phi\left(z^{(1)}_{n,\theta}(x)\right).$$

$z^{(2)}_{n,\theta}(x)$ and $z^{(1)}_{n,\theta}(x)$ are $-(n^{1/2}\theta - x)/2 \pm \sqrt{((n^{1/2}\theta + x)/2)^2 + n\mu_n^2}$.

Φ and ϕ the cdf and pdf of $N(0,1)$, resp.

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$dF_{n,\theta}$ is given by

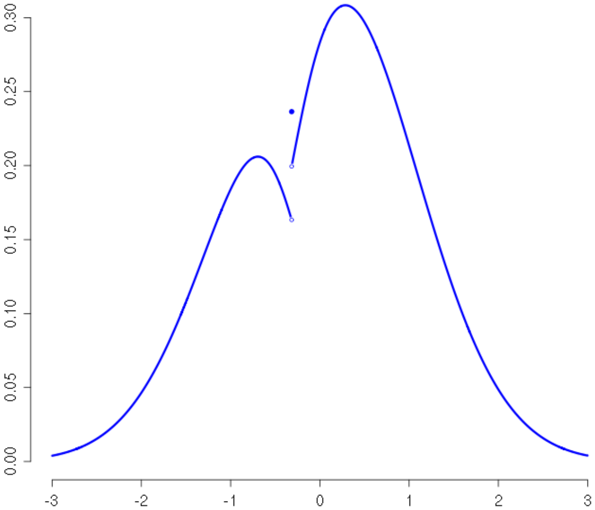
$$\begin{aligned} &\{\, \Phi(n^{1/2}(-\theta + \mu_n)) \;-\; \Phi(n^{1/2}(-\theta - \mu_n)) \,\}\; d\delta_{-n^{1/2}\theta}(x) + \\ &0.5 \times \{\, \mathbf{1}(n^{1/2}\theta + x > 0)\, \phi\left(z^{(2)}_{n,\theta}(x)\right)\, (1 + t_{n,\theta}(x)) + \\ &\qquad \mathbf{1}(n^{1/2}\theta + x < 0)\, \phi\left(z^{(1)}_{n,\theta}(x)\right)\, (1 - t_{n,\theta}(x)) \,\}\; dx \end{aligned}$$

where $t_{n,\theta}(x) := \left(((n^{1/2}\theta + x)/2)^2 + n\mu_n^2\right)^{-1/2}$.

Φ and ϕ the cdf and pdf of $N(0,1)$, resp.

The finite-sample distribution of $\hat{\theta}_{\text{AL}}$

$n = 40$, $\theta = 0.05$, $\mu_n = 0.05$

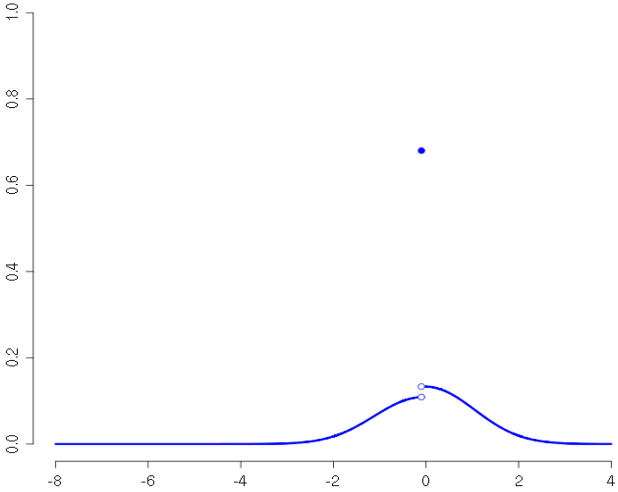


Non-normality??

- Finite-sample distribution is highly non-normal.
- Oracle property predicts normality (asymptotically).

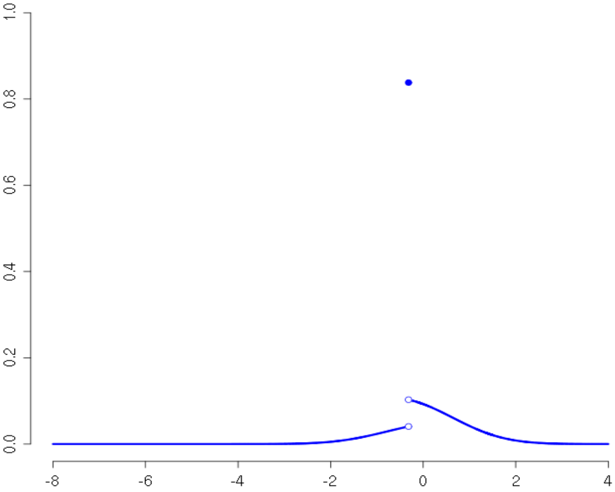
The Oracle (fixed-parameter asymptotics)

$n = 1, \qquad \mu_n = n^{-1/3}$ (consistent case)



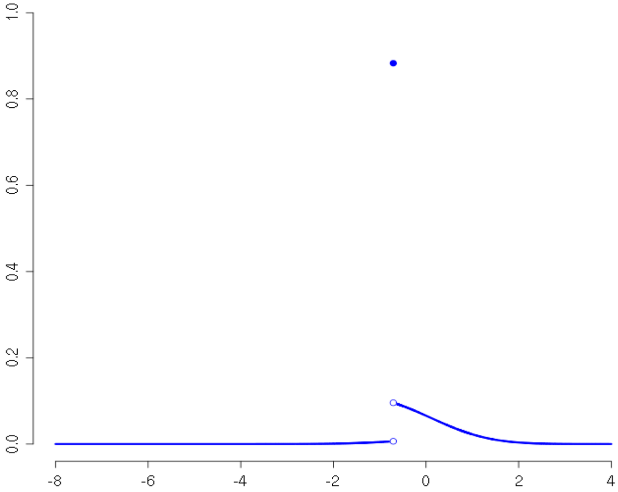
The Oracle (fixed-parameter asymptotics)

$n = 10, \qquad \mu_n = n^{-1/3}$ (consistent case)



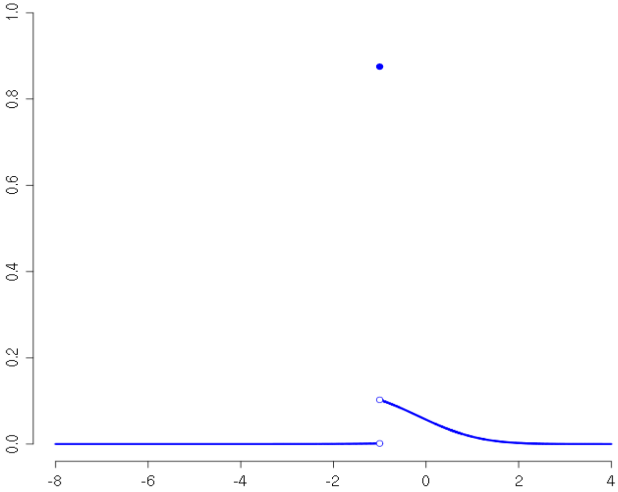
The Oracle (fixed-parameter asymptotics)

$n = 50, \qquad \mu_n = n^{-1/3}$ (consistent case)



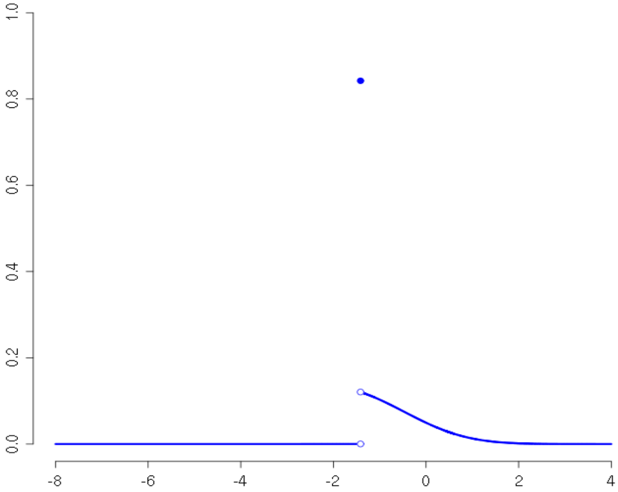
The Oracle (fixed-parameter asymptotics)

$n = 100, \qquad \mu_n = n^{-1/3}$ (consistent case)



The Oracle (fixed-parameter asymptotics)

$n = 200, \qquad \mu_n = n^{-1/3}$ (consistent case)



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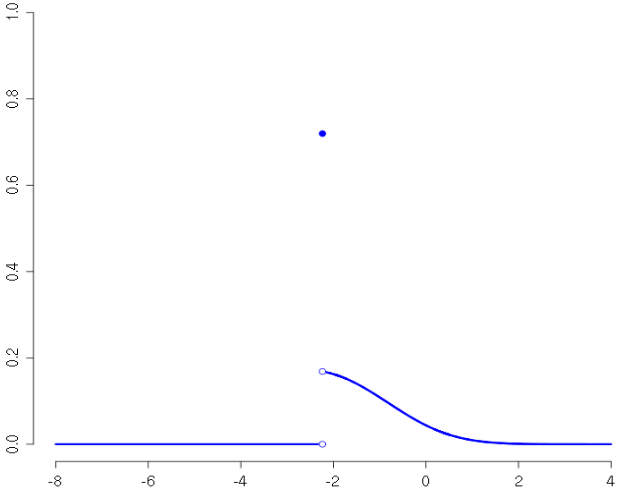
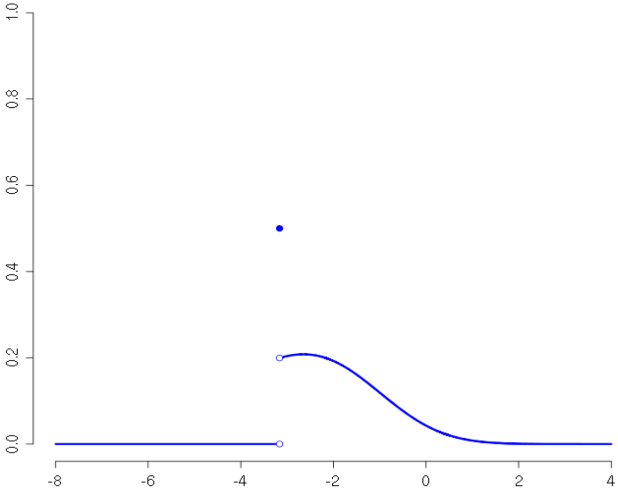
$n = 500, \qquad \mu_n = n^{-1/3}$ (consistent case)

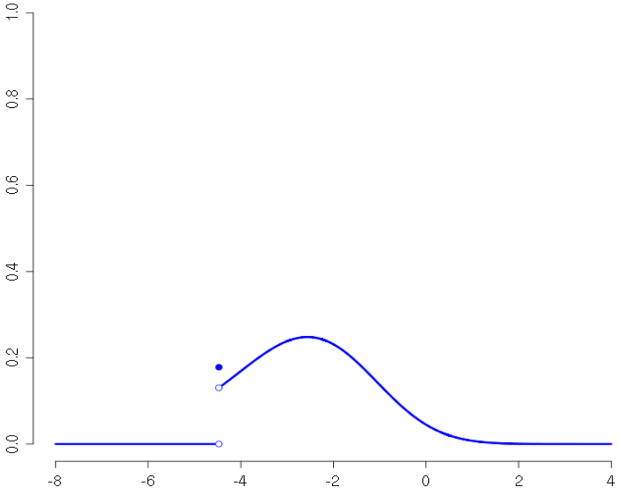


$n = 1000, \quad \mu_n = n^{-1/3}$ (consistent case)

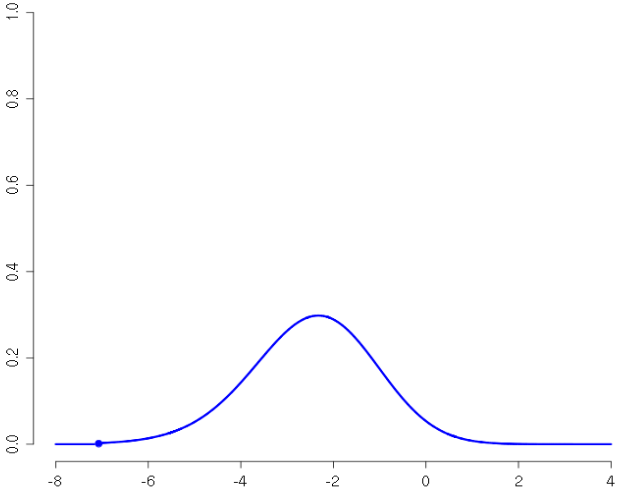


The Oracle (fixed-parameter asymptotics)

$n = 2000, \quad \mu_n = n^{-1/3}$ (consistent case)



$n = 5000, \quad \mu_n = n^{-1/3}$ (consistent case)



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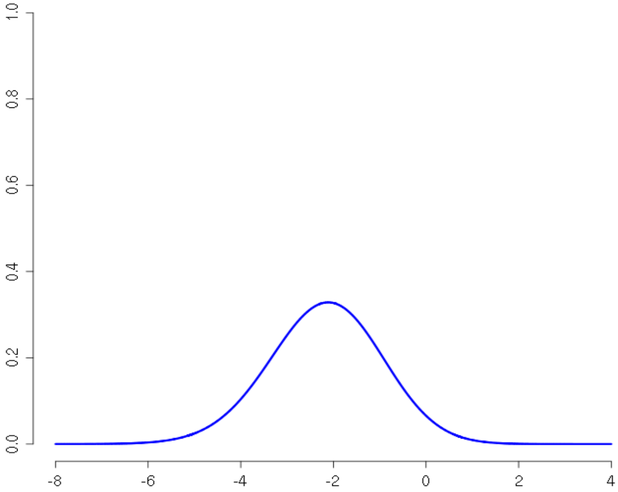
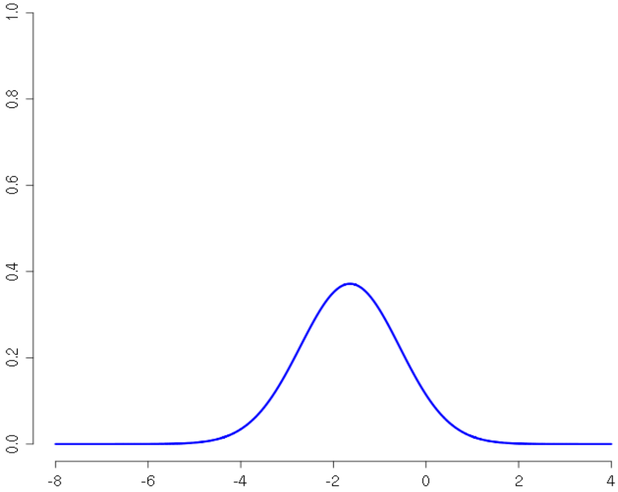


The Oracle (fixed-parameter asymptotics)

$n = 5 \times 10^4,\ \mu_n = n^{-1/3}$ (consistent case)



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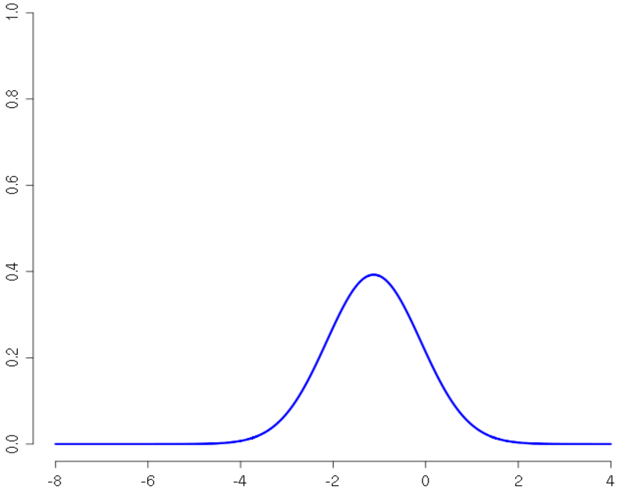
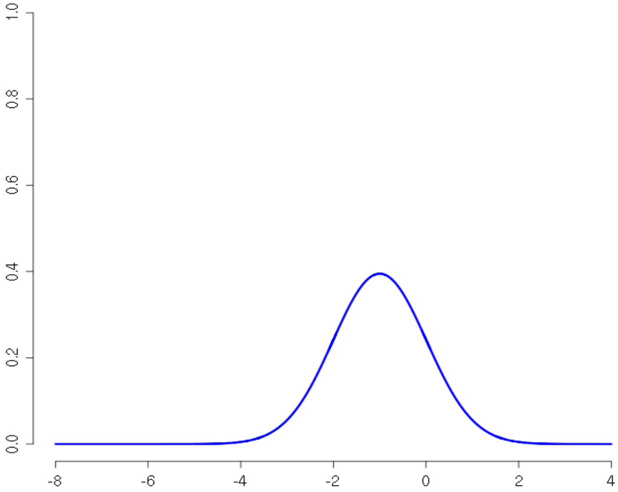


The Oracle (fixed-parameter asymptotics)

$n = 10^6, \qquad \mu_n = n^{-1/3}$ (consistent case)



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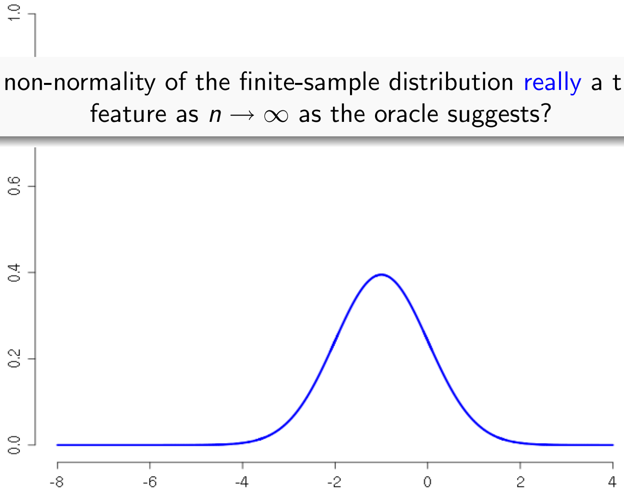

Is the non-normality of the finite-sample distribution really a transient feature as $n \to \infty$ as the oracle suggests?

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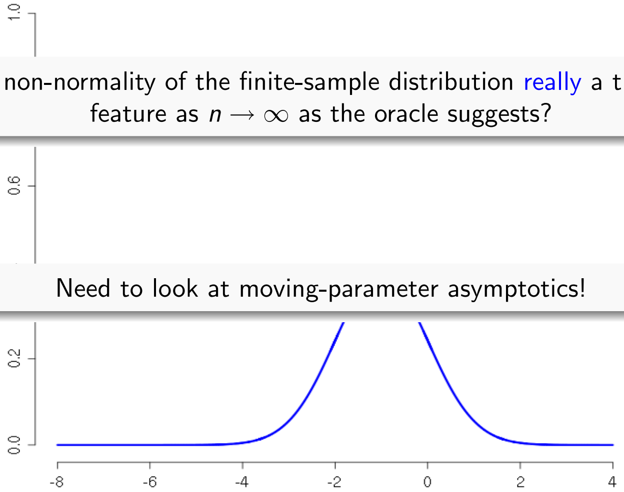

Is the non-normality of the finite-sample distribution really a transient feature as $n \to \infty$ as the oracle suggests?

Need to look at moving-parameter asymptotics!

Moving-parameter asymptotics?

Let underlying parameter θ depend on sample size:

Let $\theta_n \in \mathbb{R}$ be arbitrary, subject only to $\theta_n/\mu_n \to \zeta \in \mathbb{R} \cup \{-\infty, \infty\}$ and $n^{1/2}\theta_n \to \nu \in \mathbb{R} \cup \{-\infty, \infty\}$.

This is not really a restriction since every subsequence of θ_n contains a further subsequence with these properties. Also note that $\zeta \neq 0$ implies $\nu = \pm\infty$.

② Consistent case.

Let $\mu_n \to 0$ and $n^{1/2}\mu_n \to \infty$. Suppose the true parameter $\theta_n \in \mathbb{R}$ satisfies $\theta_n/\mu_n \to \zeta \in \mathbb{R} \cup \{-\infty, \infty\}$ and $n^{1/2}\theta_n \to \nu \in \mathbb{R} \cup \{-\infty, \infty\}$. Then F_{n,θ_n} converges weakly to

- If $0 \leq |\zeta| < \infty$: pointmass at $-\nu$
- If $|\zeta| = \infty$: $\Phi(. + \rho/\theta)$ where $n^{1/2}\mu_n^2 \to \rho$.

Depending on ζ, ν and ρ, three possible limits arise.

- Distribution collapses at a point.
- Total mass escapes to $\pm\infty$.
- Limit distribution is normal (possibly shifted!).

Moving-parameter asymptotics 2

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Moving-parameter asymptotics ②

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Non-normality persists!!

Illustration: collapsing to pointmass

Example 1: $n = 1,\qquad \zeta = 0,\ \nu = 2\qquad (\mu_n = n^{-1/3},\ \theta_n = 2n^{-1/2})$

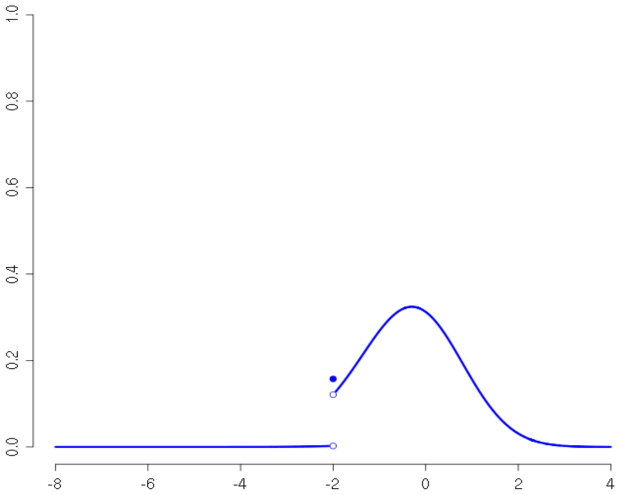


Illustration: collapsing to pointmass

Example 1: $n = 10, \qquad \zeta = 0,\ \nu = 2 \qquad (\mu_n = n^{-1/3},\ \theta_n = 2n^{-1/2})$

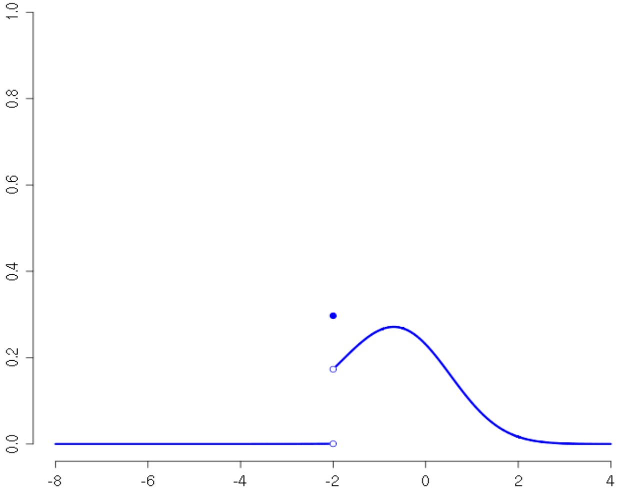


Illustration: collapsing to pointmass

Example 1: $n = 50$, $\zeta = 0,\ \nu = 2$ $(\mu_n = n^{-1/3},\ \theta_n = 2n^{-1/2})$

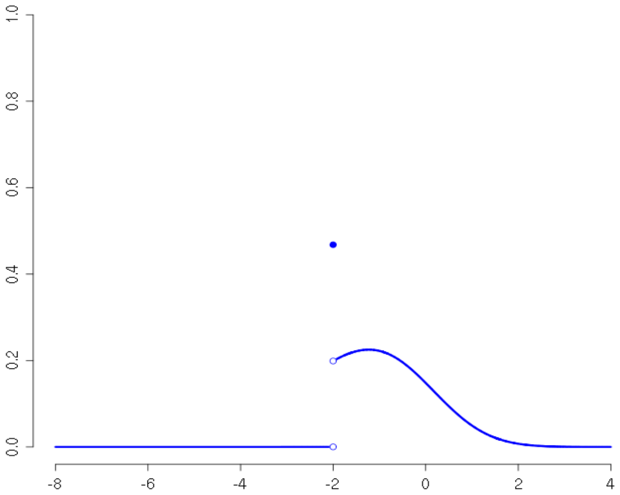
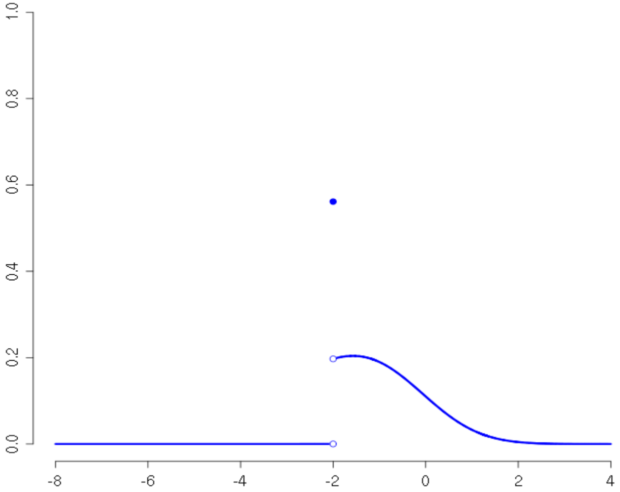


Illustration: collapsing to pointmass

Example 1: $n = 100$, $\zeta = 0,\ \nu = 2$ $(\mu_n = n^{-1/3},\ \theta_n = 2n^{-1/2})$



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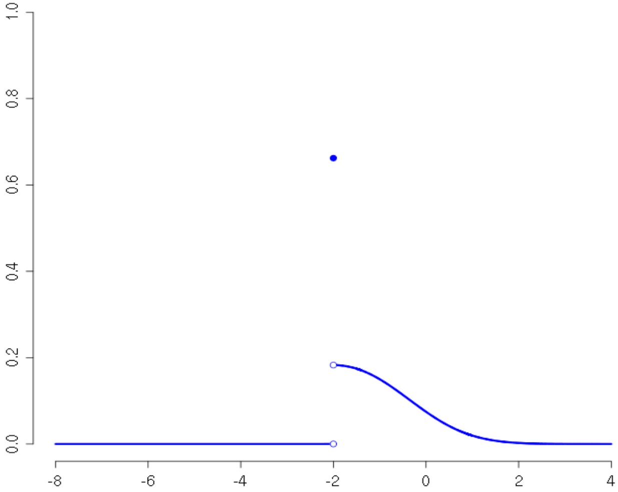

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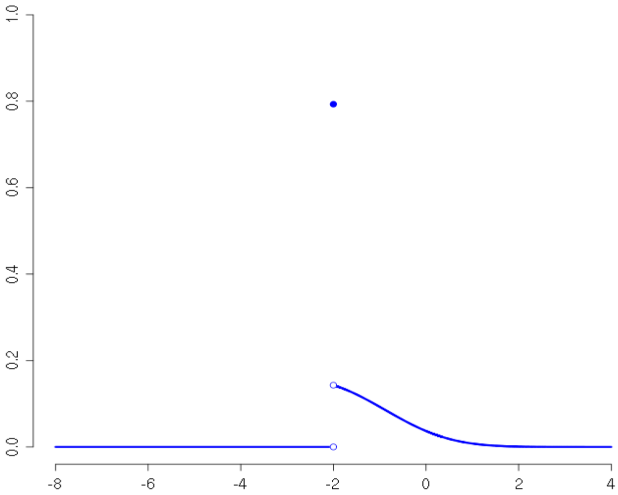

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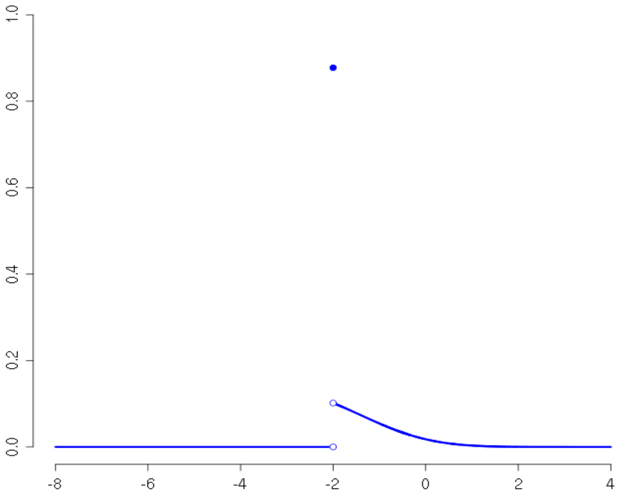

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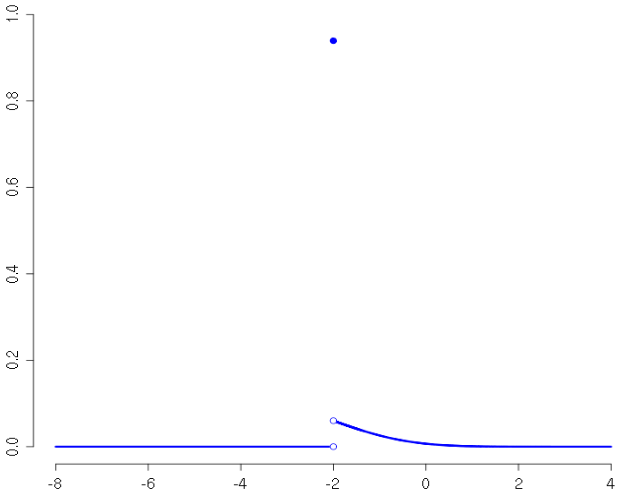

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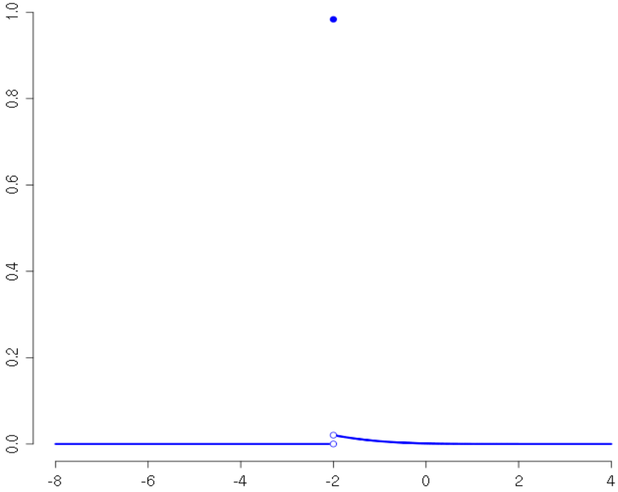

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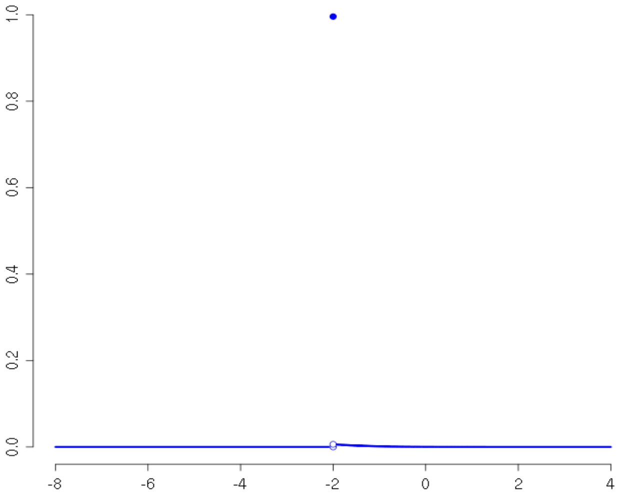

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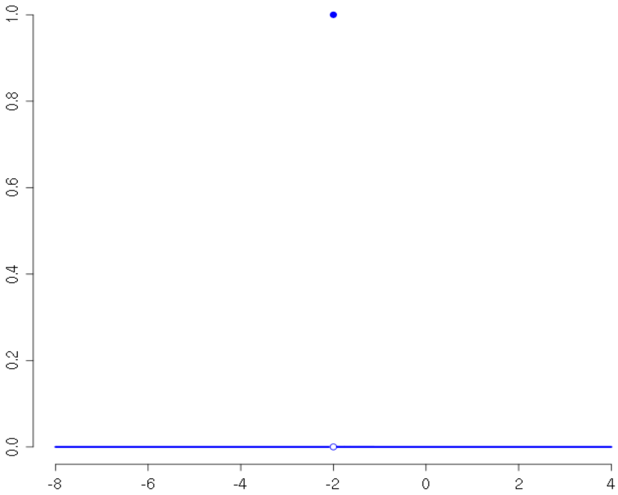

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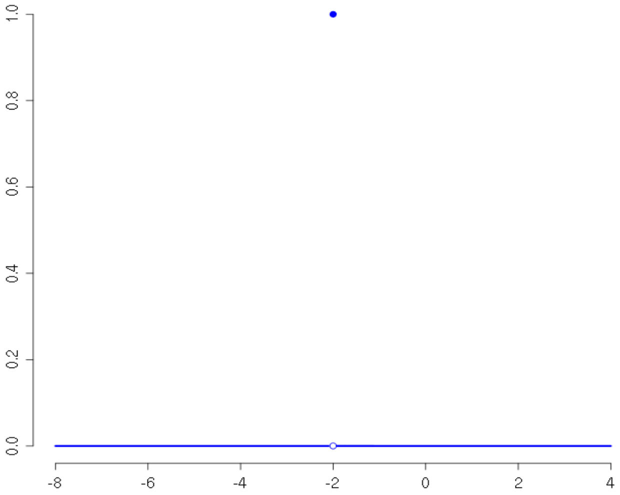
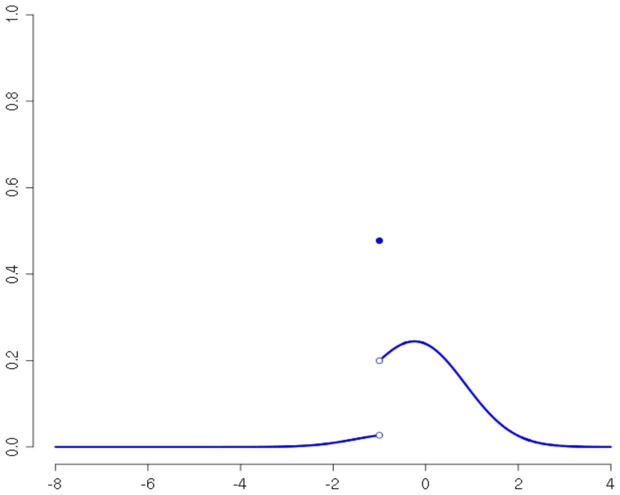


END

Illustration: mass escaping to $-\infty$

Example 2: $n = 1, \qquad \zeta = 1, \ \nu = \infty \qquad (\mu_n = n^{-1/5}, \ \theta_n = n^{-1/5})$



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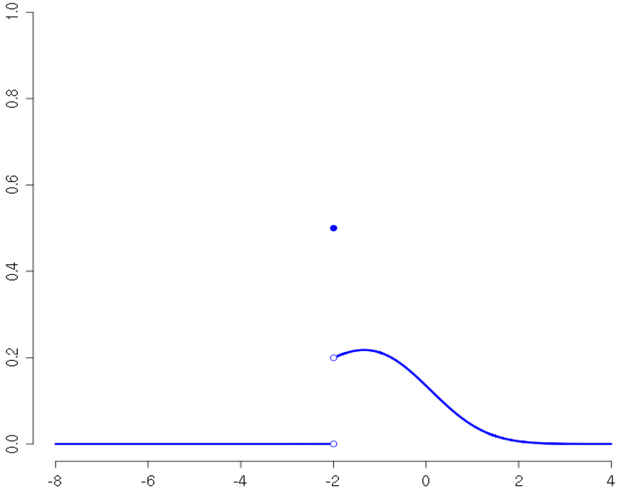

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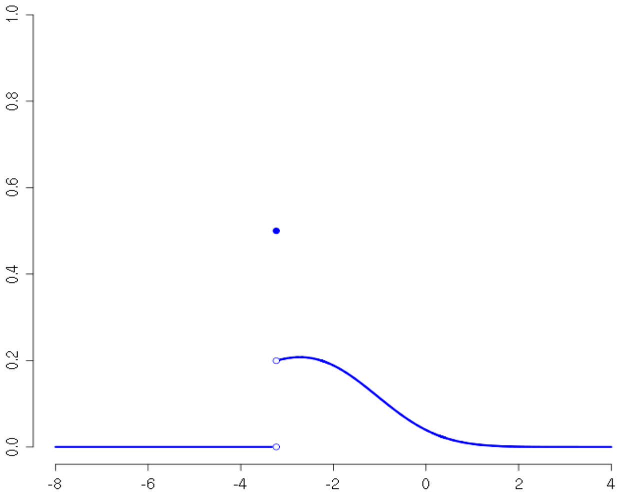

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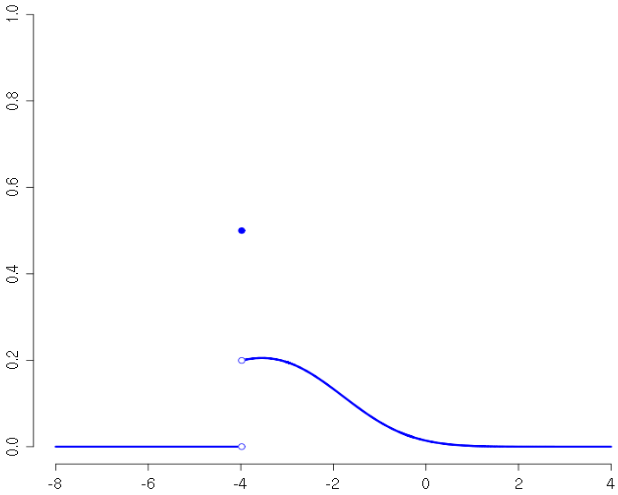

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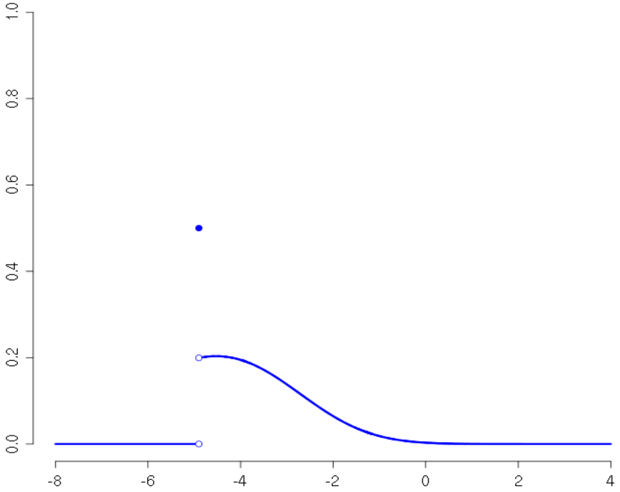

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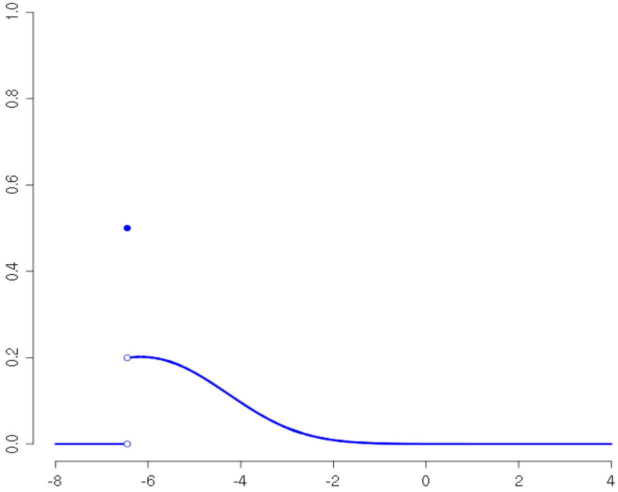

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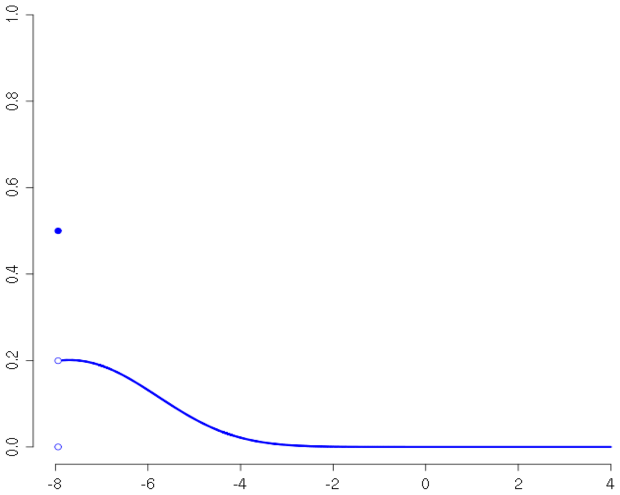

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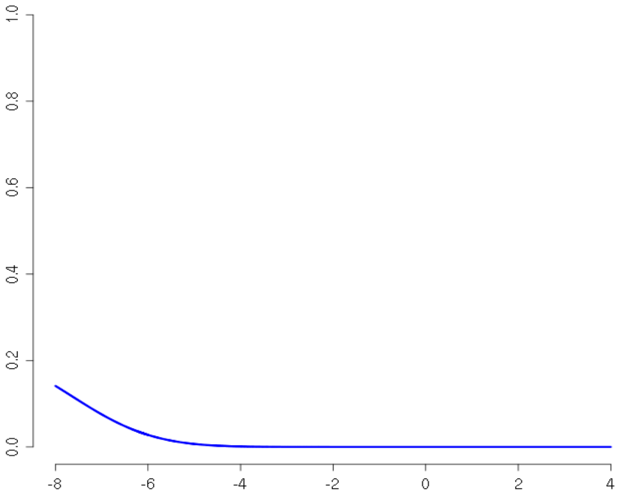

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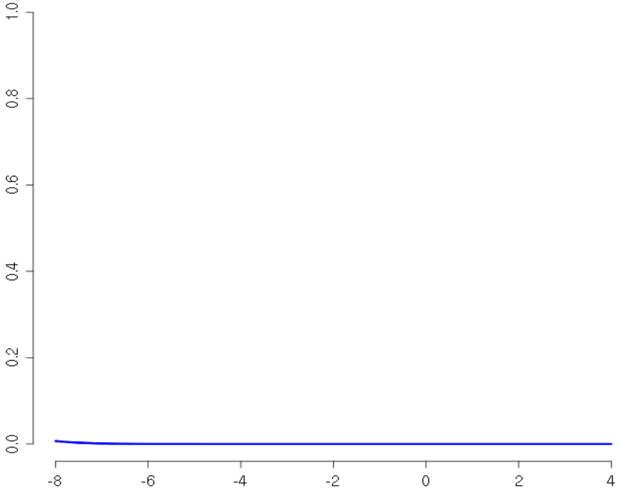

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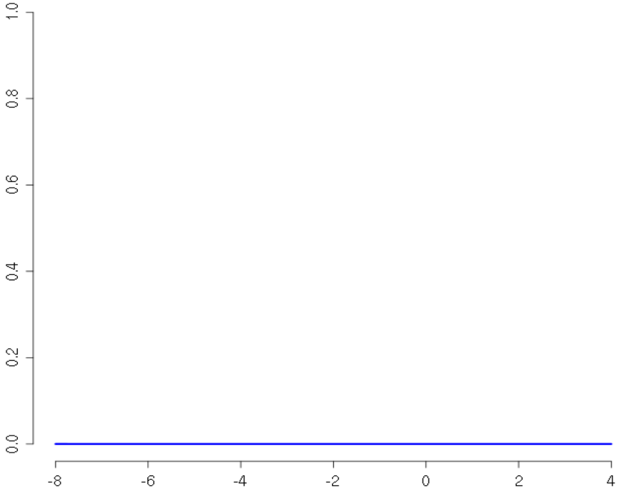

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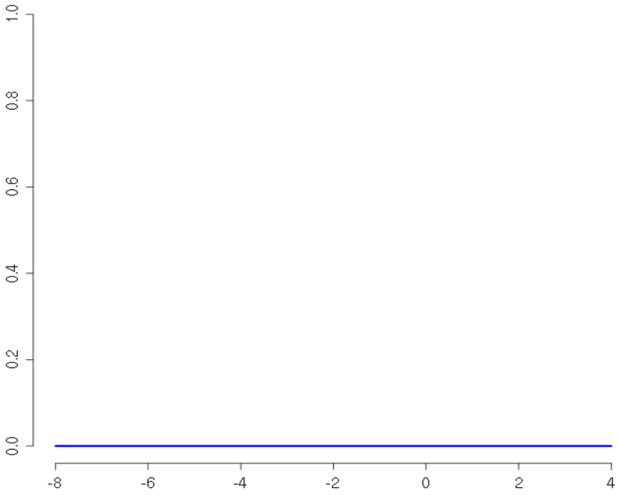

END

Moving-parameter asymptotics ❷ (cont'd)

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Zou (pointwise case) ? Above theorem implies that

$$F_{n,\theta}(x) \to \begin{cases} \mathbf{1}(x \geq 0) & \theta = 0 \quad (\implies \zeta, \nu = 0) \\ \Phi(x + \rho/\theta) & \theta \neq 0 \quad (\implies |\zeta| = \infty) \end{cases}$$

Remark: $\rho = 0 \iff n^{1/4}\mu_n \to 0$.

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Remark: $\rho = 0 \iff n^{1/4}\mu_n \to 0$.

Uniform consistency and alternative scaling ❷

- Adaptive LASSO has in a uniform sense a rate of convergence that is slower than $n^{1/2}$.
- The "correct" uniform rate can be shown to be μ_n^{-1}.
- In a moving-parameter framework, the asymptotic distribution of $\mu_n^{-1}(\hat{\theta}_{\text{AL}} - \theta)$ collapses to pointmass.

Let $\mu_n \to 0$ and $n^{1/2}\mu_n \to \infty$. Suppose the true parameter $\theta_n \in \mathbb{R}$ satisfies $\theta_n/\mu_n \to \zeta \in \mathbb{R} \cup \{-\infty, \infty\}$. Then $G_{n,\theta_n} := P(\mu_n^{-1}(\hat{\theta}_{\text{AL}} - \theta) \le x)$ converges weakly to

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Moving parameter asymptotic ② "conclusion"

Above theorems reflect that

$$\hat{\theta}_{\mathsf{AL}} - \theta = \text{“BIAS”} + \text{“FLUCTUATION”}$$

where

- "BIAS" is $O(n^{-1/2})$ in a pointwise sense but is only $O(\mu_n)$ in a uniform sense, whereas
- "FLUCTUATION" is always of order $n^{-1/2}$.

Moving parameter asymptotics

1. Conservative case.

Let $\mu_n \to 0$ and $n^{1/2}\mu_n \to m$, $0 \le m < \infty$. Suppose the true parameter $\theta_n \in \mathbb{R}$ satisfies $n^{1/2}\theta_n \to \nu \in \mathbb{R} \cup \{-\infty, \infty\}$. Then F_{n,θ_n} converges weakly to

- If $\nu \in \mathbb{R}$

$$\mathbf{1}(\nu + x \ge 0)\,\Phi\left(-(\nu - x)/2 + \sqrt{((\nu + x)/2)^2 + m^2}\right) + \mathbf{1}(\nu + x < 0)\,\Phi\left(-(\nu - x)/2 - \sqrt{((\nu + x)/2)^2 + m^2}\right)$$

- $\Phi(x)$ if $|\nu| = \infty$.

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Note: Asymptotic distributions are the same as finite-sample distribution, except that $n^{1/2}\theta_n$ and $n^{1/2}\mu_n$ have settled down to their limiting values, capturing finite-sample behavior very well.

Moving parameter asymptotics ❶ (cont'd)

- $\hat{\theta}_{\text{AL}}$ is now uniformly $n^{1/2}$-consistent.
- Fixed-parameter asymptotics: previous theorem implies that $F_{n,\theta}(x)$ converges to
  - $\mathbf{1}(x \geq 0)\,\Phi\left(\frac{x}{2} + \sqrt{(\frac{x}{2})^2 + m^2}\right) + \mathbf{1}(x < 0)\,\Phi\left(\frac{x}{2} - \sqrt{(\frac{x}{2})^2 + m^2}\right)$
    if $\theta = 0$ $(\nu = 0)$
  - $\Phi(x)$ if $\theta \neq 0$ $(|\nu| = \infty)$
- Fixed-parameter asymptotic distributions are also non-normal, capturing behavior the finite-sample distributions to some extent (no oracle here).

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Other PLSEs

Results are similar for hard-thresholding, soft-thresholding (LASSO), and SCAD estimator. (Pötscher & Leeb, 2007).

- Identical results in terms of (uniform) consistency.
- Analogous (asymptotic) distributional results.

Confidence sets based on the adaptive LASSO

Let $C_n = [\hat{\theta}_{\text{AL}} - a_n, \hat{\theta}_{\text{AL}} + b_n]$.

The infimal coverage probability $\inf_{\theta \in \mathbb{R}} P_{n,\theta}(\hat{\theta}_{\text{AL}} \in C_n)$ is given by

$$\Phi(n^{1/2}(a_n - \mu_n)) - \Phi\left(n^{1/2}((a_n - b_n)/2 - \sqrt{((a_n + b_n)/2)^2 + \mu_n^2}\right)$$

if $a_n \leq b_n$ and

$$\Phi\left(n^{1/2}((a_n - b_n)/2 + \sqrt{((a_n + b_n)/2)^2 + \mu_n^2})\right) - \Phi(n^{1/2}(-b_n + \mu_n))$$

if $a_n > b_n$.

Symmetric intervals ($a_n = b_n$) can be shown to be the shortest ones for a given infimal coverage probability δ.

Confidence sets based on PLSEs

- For each $n \in \mathbb{N}$, we have

$$a_{n,H} > a_{n,\mathrm{AL}} > a_{n,L} > a_{n,\mathrm{OLS}} \quad \text{for a given } \delta > 0$$

- Asymptotically, the following holds.

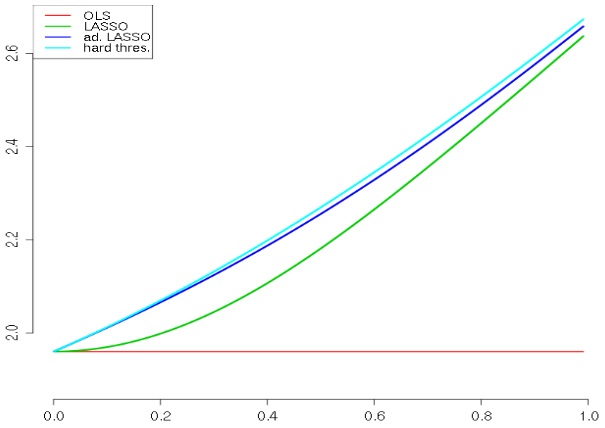
1. Conservative case. All quantities are of the same order $n^{-1/2}$.

$$a_{n,H} \sim a_{n,\mathrm{AL}} \sim a_{n,L} \sim a_{\mathrm{OLS}}$$

2. Consistent case. $a_{n,H}, a_{n,L}$, and $a_{n,A}$ are one order of magnitude larger than $a_{n,\mathrm{OLS}}$.

$$a_H / a_{\mathrm{OLS}} \sim a_{\mathrm{AL}} / a_{\mathrm{OLS}} \sim a_L / a_{\mathrm{OLS}} \sim n^{1/2} \mu_n \to \infty$$

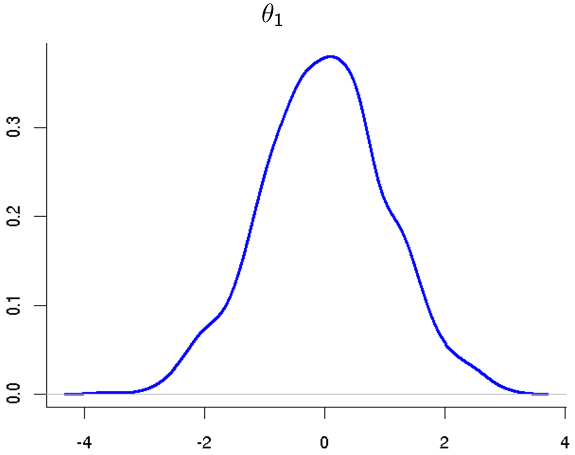
Plot of $n^{1/2}a_n$ against $n^{1/2}\mu_n$ for $\delta = 0.95$.



Simulations - remove orthogonality assumption

$k = 4$, $n = 200$, $\theta = (3, 1.5, 0, 0)' + 2/n^{1/2}(0, 0, 1, 1)'$, $X'X = n\Omega$ with $\Omega_{ij} = 0.5^{|i-j|}$, 1000 simulations

- $\mu_n = n^{-1/3}$



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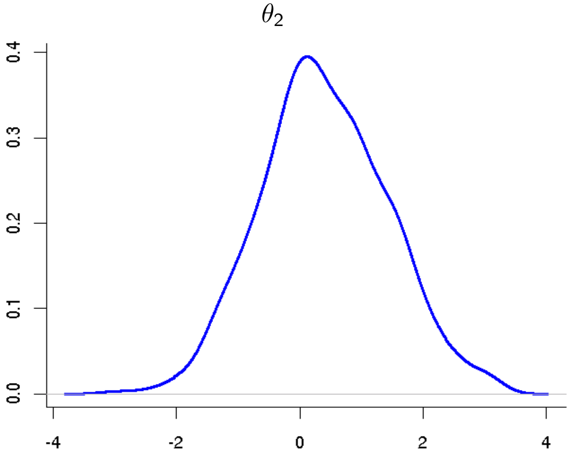

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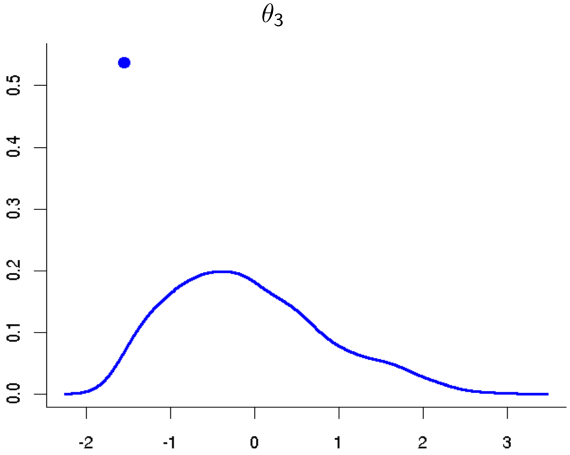

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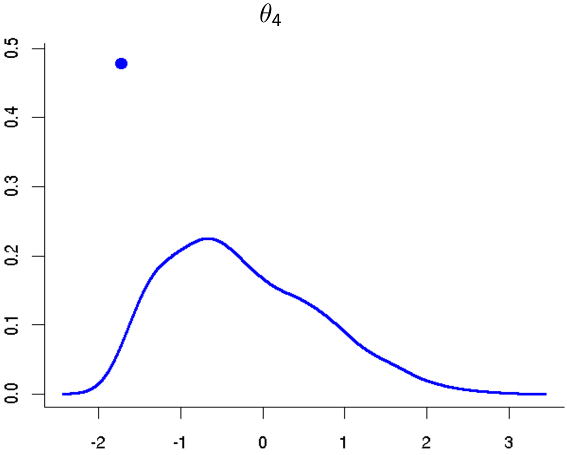

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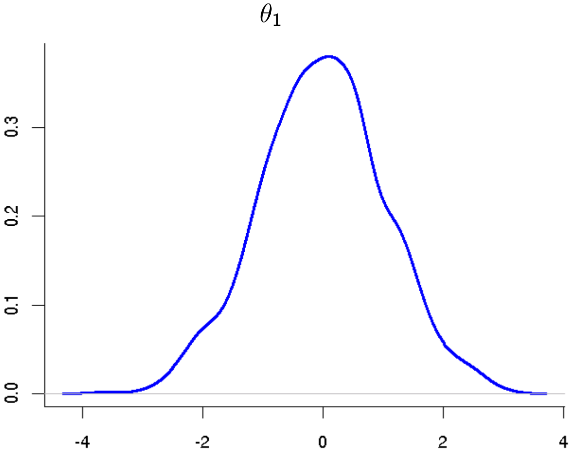
- Choose μ_n through cross-validation.



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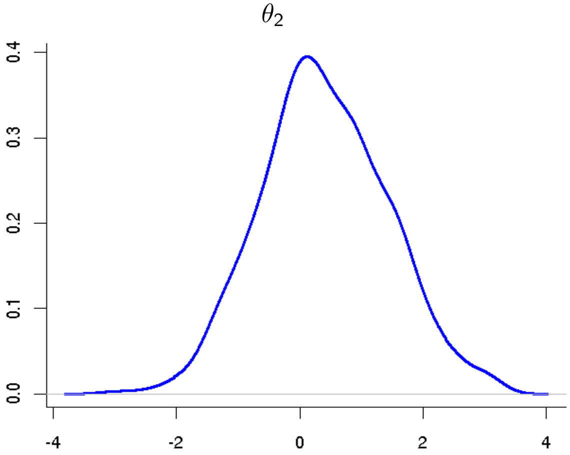

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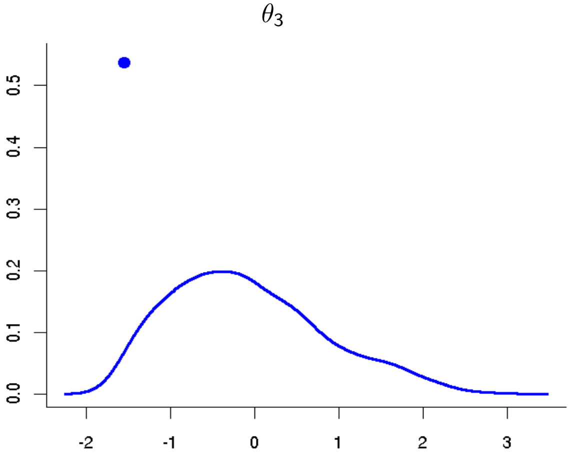

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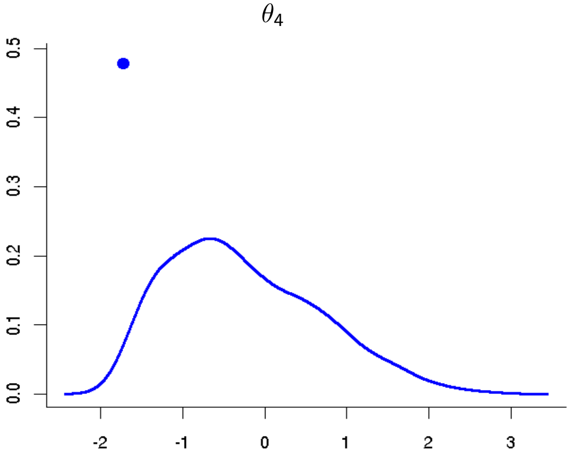

Estimation of the cdf of $n^{1/2}(\hat{\theta}_{\text{AL}} - \theta)$?

Let $F_{n,\theta}$ be the distribution function of $n^{1/2}(\hat{\theta}_{\text{AL}} - \theta)$.

Let $\mu_n \to 0$ and $n^{1/2}\mu_n \to m$ with $0 \le m \le \infty$. Then *every consistent* estimator $\hat{F}_n(t)$ of $F_{n,\theta}(t)$ satisfies

$$\lim_{n\to\infty} \sup_{|\theta|<c/n^{1/2}} P_{n,\theta}\left(\left|\hat{F}_n(t) - F_{n,\theta}(t)\right| > \varepsilon\right) \geq \frac{1}{2}$$

for each $\varepsilon < (\Phi(t+m) - \Phi(t-m))/2$ and each $c > |t|$.

In particular, not uniformly consistent estimator for $F_{n,\theta}(t)$ exisits!

Analogous result for cdf under μ^{-1}-scaling.

Proof rests on Pötscher & Leeb (2006).

Estimation of the cdf of $n^{1/2}(\hat{\theta}_{\text{AL}} - \theta)$?

Finite-sample result:

Let $\mu_n \to 0$ and $n^{1/2}\mu_n \to m$ with $0 \le m \le \infty$. Then *every* estimator $\hat{F}_n(t)$ of $F_{n,\theta}(t)$ satisfies

$$\sup_{|\theta|<c/n^{1/2}} P_{n,\theta}\left(\left|\hat{F}_n(t) - F_{n,\theta}(t)\right| > \varepsilon\right) \ \geq \ \frac{1}{2}$$

for each $\varepsilon < (\Phi(t+m) - \Phi(t-m))/2$, for each $c > |t|$ and each sample size n. Hence

$$\liminf_{n\to\infty} \inf_{\hat{F}_n(t)} \sup_{|\theta|<c/n^{1/2}} P_{n,\theta}\left(\left|\hat{F}_n(t) - F_{n,\theta}(t)\right| > \varepsilon\right) \ = \ 1$$

for each $\varepsilon < (\Phi(t+m) - \Phi(t-m))/2$ and each $c > |t|$ where the infimum extend over *all* estimators $\hat{F}_n(t)$.

Conclusions

- The finite-sample distribution of the adaptive LASSO estimator and other PLSEs are highly non-normal.
- Non-normality persists in large samples. This can be seen through a "moving-parameter" asymptotic framework.
- Fixed-parameter asymptotics (as underlying the oracle-property) paint a misleading picture of the performance of the estimator due to the non-uniformity of these results.
- Confidence intervals in the consistent case are larger by one order of magnitude compared to the unpenalized estimator.
- The distribution function of the adaptive LASSO estimator and other PLSEs cannot be estimated in a uniformly consistent manner.
- NOT a critisim on PLSEs per se, but relying on fixed-parameter asymptotics in this context is dangerous.

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