Introduction to Ordinal Analysis

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Part I
Handouts

1 A brief reminder

1.1 Abstract structures and logical inferences

In a very general setting mathematics is concerned with the study of abstract structures. An abstract structure has the form $\mathfrak{M} = (M, \mathscr{C}, \mathscr{R}, \mathscr{F})$ where M is a non–void set, $\mathscr{C}$ as subset of M, $\mathscr{R}$ a set of relations on M and $\mathscr{F}$ a set of functions on M. Associated to an abstract structure is its abstract language $\mathscr{L}_{\mathfrak{M}} = \mathscr{L}(\mathcal{C}, \mathcal{R}, \mathcal{F})$ which comprises a set $\mathcal{C}$ of constants for elements of M, a set $\mathcal{R}$ of symbols for the relations in $\mathscr{R}$ and a set $\mathcal{F}$ of symbols for the functions in $\mathscr{F}$.

In general a *signature* for a logical language is a triple $(\mathcal{C}, \mathcal{R}, \mathcal{F})$ where every $R \in \mathcal{R}$ and $f \in \mathcal{F}$ carries its arity $0 < \#R \in \mathbb{N}$ and $0 < \#f \in \mathbb{N}$.

An abstract structure $\mathfrak{M} = (M, \mathscr{C}, \mathscr{R}, \mathscr{F})$ *interprets* a signature $(\mathcal{C}, \mathcal{R}, \mathcal{F})$ if every *constant* $c \in \mathcal{C}$ has an interpretation $c^{\mathfrak{M}} \in M$, every *relation symbol* $R \in \mathcal{R}$ an interpretation $R^{\mathfrak{M}} \subseteq M^{\#R}$ and every *function symbol* $f \in \mathcal{F}$ an interpretation $f^{\mathfrak{M}} \colon M^{\#f} \longrightarrow M$.

We say that a signature *matches* the structure $\mathfrak{M}$ if there is a symbol $c \in \mathcal{C}$, $R \in \mathcal{R}$ and f in $\mathcal{F}$ for every constant in $\mathscr{C}$, every relation in $\mathscr{R}$ and every function in $\mathscr{F}$.

The *closed terms* of a signature $(\mathcal{C}, \mathcal{R}, \mathcal{F})$ are either constants or composed terms of the form $(ft_1, \ldots, t_n)$ where $\#f = n$ and $t_1, \ldots, t_n$ are constants or previously defined composed terms.

Atomic sentences have the form $(Rt_1, \ldots, t_n)$ where R is an n–ary relation symbols and $t_1, \ldots, t_n$ are closed terms. Starting from atomic sentences we can inductively build a logical language $\mathscr{L}(\mathcal{C}, \mathcal{R}, \mathcal{F})$ using the familiar boolean operations and quantifications. If quantification is restricted to individuals we talk about a *first order logic* $\mathscr{L}_1(\mathcal{C}, \mathcal{R}.\mathcal{F})$. If we also allow quantifiers ranging over relations we talk about a *second order logic* $\mathscr{L}_2(\mathcal{C}, \mathcal{R}, \mathcal{F})$.

For an abstract structure $\mathfrak{M}$ that interprets the signature of a logical language every closed term t possesses a canonical interpretation $t^{\mathfrak{M}} \in M$.

Defining $\mathfrak{M} \models (Rt_1, \ldots, t_n)$ iff $(t_1^{\mathfrak{M}}, \ldots, t_n^{\mathfrak{M}}) \in R^{\mathfrak{M}}$ and continuing inductivley according to the meaning of the logical operations we obtain a canonical *satisfiability relation* $\mathfrak{M} \models F$ for the sentences F in the language $\mathscr{L}(\mathcal{C}, \mathcal{R}, \mathcal{F})$. We say that F is *valid in* $\mathfrak{M}$ iff $\mathfrak{M} \models F$.

If $S \cup \{F\}$ is a set of $\mathscr{L}(\mathcal{C}, \mathcal{R}, \mathcal{F})$–sentences we call $S \models F$ a *logical inference* iff for every abstract structure $\mathfrak{M}$ that interprets $(\mathcal{C}, \mathcal{R}, \mathcal{F})$ the fact that $\mathfrak{M} \models G$ holds true for all $G \in S$ also implies $\mathfrak{M} \models F$.

An abstract structure $\mathfrak{M}$ *satisfies* a set S of $\mathscr{L}(\mathcal{C}, \mathcal{R}, \mathcal{F})$–sentences iff $\mathfrak{M}$ interprets $(\mathcal{C}, \mathcal{R}, \mathcal{F})$ and satisfies all sentences in S.

A set S of $\mathscr{L}(\mathcal{C}, \mathcal{R}, \mathcal{F})$–sentences is *consistent* iff there is a structure $\mathfrak{M}$ which satifies S.

A set S of $\mathscr{L}(\mathcal{C}, \mathcal{R}, \mathcal{F})$–sentences is *logically valid* iff every structure which interprets $(\mathcal{C}, \mathcal{R}, \mathcal{F})$ satifies S.

1.1 Exercise Give a formal definition of $t^{\mathfrak{M}}$ and $\mathfrak{M} \models F$.

1.2 Exercise Show that $S \models F$ iff $S \cup \{F\}$ is inconsistent.

1.3 Exercise Show that a set S of $\mathscr{L}(\mathcal{C}, \mathcal{R}, \mathcal{F})$–sentences is consistent iff there is no $\mathscr{L}(\mathcal{C}, \mathcal{R}, \mathcal{F})$–sentence F such that $S \models F \wedge \neg F$.

1.2 Formal derivations

We extend a language $\mathscr{L} := \mathscr{L}(\mathcal{C}, \mathcal{R}, \mathcal{F})$ by adding free individual variables, denoted by $x, y, z, x_1, \ldots$, and free relation variables, denoted by $X, Y, Z, X_1, \ldots$, together with their arities $\#X \in \mathbb{N}$. In forming terms individual variables are treated like constants; in forming formulae relation variables are treated like relation symbols. Terms without occurrences of free individual variables are *closed*, sentences are formulae in which neither individual variables nor relation variables occur freely.

Even if a structure $\mathfrak{M}$ interprets $\mathscr{L}$ there is no canonical interpretation for terms containing free individual variables and no canonical satisfaction relation for formulae containing free variables. Interpretation and satisfaction of terms and formulae need an *assignment* Φ which assigns an element $\Phi(x) \in M$ to every free individual variable x

and a set $\Phi(X) \subseteq M^{\#X}$ to every relation variable X. We denote by $\mathfrak{M} \models F[\Phi]$ that $\mathfrak{M}$ satisfies the formula F under the assignment Φ.

We extend the definition of a logical inference to sets of $\mathscr{L}$–formulae.

1.4 Definition Let $S \cup \{F\}$ be a set of $\mathscr{L}$–formulae. Then $S \models F$ is a logical inference iff for every structure $\mathfrak{M}$ that interprets $\mathscr{L}$ and every assignment Φ we have

$$\mathfrak{M} \models G[\Phi] \textit{ for all formulae in } G \in S \textit{ implies } \mathfrak{M} \models F[\Phi].$$

1.5 Definition Let $\mathscr{L} := \mathscr{L}(\mathcal{C}, \mathcal{R}, \mathcal{F})$ be a logical language. A *formal rule* is a figure

$$P_1, \ldots, P_n \vdash C,$$

where $n \geq 0$ and $\{P_1, \ldots, P_n, C\}$ is a set of $\mathscr{L}$–formulae.

A *formal system* $\mathbb{S}$ is a set of formal rules.

Given a formal system $\mathbb{S}$ we define *formal derivability* $A_1, \ldots, A_m \vdash_{\mathbb{S}} F$ inductively by:

- *If* $A_1, \ldots, A_m \vdash_{\mathbb{S}} P_i$ *for* $i = 1, \ldots, n$ *and* $P_1, \ldots, P_n \vdash F$ *is a rule of* $\mathbb{S}$ *then* $A_1, \ldots, A_m \vdash_{\mathbb{S}} F$.

A formal system $\mathbb{S}$ is *sound* if for every rule $P_1, \ldots, P_n \vdash C$ in $\mathbb{S}$ we have $P_1, \ldots, P_n \models C$.

1.6 Exercise (Soundness Theorem) Let $\mathbb{S}$ be a sound formal system. Show that $A_1, \ldots, A_n \vdash_{\mathbb{S}} F$ entails $A_1, \ldots, A_n \models F$.

The following completeness theorem by Kurt Gödel is one of the most important theorems of Mathematical Logic.

1.7 Theorem *(Gödel's completeness theorem) Let* $\mathscr{L}_1$ *be a first order language and* $S \cup \{F\}$ *a set of* $\mathscr{L}_1$*–formulae. Then there is a sound formal system* $\mathbb{S}$ *such that* $S \models F$ *entails* $S_1, \ldots, S_n \vdash_{\mathbb{S}} F$ *for a finite subset* $\{S_1, \ldots, S_n\}$ *of* S.

There is a fraternal twin to Gödel's completeness theorem.

1.8 Theorem *(Compactness Theorem) Let* $\mathscr{L}_1$ *be a first order language and* S *a set of* $\mathscr{L}_1$*–sentences. If every finite subset* $S_i \subseteq S$ *is consistent, then* S *is consistent.*

1.9 Exercise Show that Gödel's completeness theorem entails the compactness theorem. (The opposite direction—though true—is much harder to show).

A formal dervation $A_1, \ldots, A_n \vdash_{\mathbb{S}} F$ in a formal system $\mathbb{S}$ can be viewed as a finite tree whose root is labelled by F, whose leaves are labelled by the formulae A_i and which is is locally correct with respect to the rules in $\mathbb{S}$. This makes the correctness of a formal proof machine–checkable, i.e., decidable. Admittedly in practice mathematical proofs are not formalized to the point that they become machine–checkable, but they should be formalizable in principle. This fact is responible for the intersubjectibility of mathematical proofs.

For full second order logic there is no compactness theorem, hence also no completeness theorem. So full second order logic is, in principle, useless for mathematical reasoning. Nevertheless there are sound formal systems for second order logic.[1]

1.3 Why ordinal analysis?

Gödel's completeness theorem establishes a tool for the investigation of abstract structures. We can try to characterize a structure $\mathfrak{M}$ by a set of first order sentences which are characteristic for and valid in $\mathfrak{M}$, the axioms for $\mathfrak{M}$. Starting from the axioms for $\mathfrak{M}$ we can argue by logical inferences to ensure that everything we conclude is a theorem of $\mathfrak{M}$. Examples for this approach are group theory, ring theory, field theory and similar algebraic disciplines.

In setting up an axiom system for a structure $\mathfrak{M}$ we are confronted with a couple of problems. First we have to ensure that the set of axioms is consistent. This causes no problems in case of groups, rings, field etc, since there are finite structures which satisfy the finitely many axioms. The second problem is that of categoricity, i.e., the question whether we can characterize the structure by the axioms up to isomorphism. This, however, is in general not possible for a first order axiom system by the compactness theorem. In the case of groups, rings etc., it is not even desirable since we know that there are many non isomorphic groups, rings, ...

The situation is different when we try to axiomatize standard structures which we believe to be familiar with. The first—and probably most important— such structure is the structure $\mathfrak{N}$ of natural numbers. We have (and all our mathematical ancestors had) in some sense a clear intuition of this structure. Here it would be desirable to have an axiomatization up to isomorphism but this is excluded by the compactness theorem.There are categorial second order axiom systems for $\mathfrak{N}$ but, according to the lack of a completeness theorem for second order logic, they are mathematically useless. So we have to resign categoricity.

There are well-established axiom system for $\mathfrak{N}$, e.g. the Peano axioms which we will later introduce in detail. Since we have resigned categoricity it remains the problem of consistency. This is not so easy to solve as in the case of the group—or similar algebraic—axioms since the standard structure $\mathfrak{N}$ is an infinite structure. Therefore any adequate axiomatization of $\mathfrak{N}$ has to incorporate infinity which entails that there exist no finite structures that satisfy these axioms (as e.g. in group theory). But $\mathfrak{N}$ is in some sense the simplest infinite structure. In order to build a structure which satisfies the axioms for $\mathfrak{N}$ we need a structure somehow above $\mathfrak{N}$ which itself needs a consistent axiomatization which then is likely to embrace the axioms for $\mathfrak{N}$.

This exposes a foundational problem. Hilbert in his programme suggested a way to solve this problem (even aiming at solving the consistency problem for all existing mathematics) by formalization. Since a formal proof is a finite figure it should be likely that we can show by finitistic—i.e. purely finite combinatorical—means that there cannot be a proof figure of a contradiction.

This hope was destroyed by Gödel's incompleteness theorems in which he showed that a proof of the consistency of any recursively enumerable axiom system for $\mathfrak{N}$ has

[1] These system, however, should rather be viewed as formal systems for a two sorted first order logic.

to exceed the means of this axiom system. Especially there cannot be a consistency proof for a recursively enumerable axiom system of $\mathfrak{N}$ by finite combinatorics.

However, despite of Gödels incompleteness theorems Gerhard Gentzen in [1] gave a consistency proof for the Peano axioms for $\mathfrak{N}$. His proof only used finitistic means except for an application of a transfinite induction along a well–ordering of order–type ε_0. By Gödel's incompleteness theorem it therefore follows that transfinite induction up to ε_0 cannot be provable from the Peano axioms. In a later paper [2] he showed that conversely any ordinal less than ε_0 can be represented by a well–ordering whose well–foundedness is provable in Peano arithmetic. This was the birth of ordinally informative proof theory. Since then we define the proof theoretic ordinal of an axiom system T as the supremum of the order–types of well–orderings which are elementarily definable in the language of T and whose well–foundedness is provable in T.

As we will see later the proof–theoretic ordinal of an axiom system in fact incorporates a measure for the *performance* of an axiom system with respect to the intended standard structure and the universe of its subsets above it.

The aim of the course is to give an introduction to ordinal analysis on the example of an axiom system for $\mathfrak{N}$ which is equivalent to the Peano axioms.

References

[1] G. GENTZEN, *Die Widerspruchsfreiheit der reinen Zahlentheorie*, **Mathematische Annalen**, vol. 112 (1936), pp. 493–565.

[2] ——, *Beweisbarkeit und Unbeweisbarkeit von Anfangsfällen der transfiniten Induktion in der reinen Zahlentheorie*, **Mathematische Annalen**, vol. 119 (1943), pp. 140–161.

These handouts will be continued during the Sommer–school.