Proof Mining

Ulrich Kohlenbach
Department of Mathematics
Technische Universität Darmstadt

Hilbert Bernays Summer School , Universität Göttingen, 25-29.7.2016

Overview of Content

Lecture I: General Introduction to the Unwinding of Proofs ('Proof Mining') and first Methods of Proof Mining.

Overview of Content

Lecture I: General Introduction to the Unwinding of Proofs ('Proof Mining') and first Methods of Proof Mining.

Lecture II: Logical Metatheorems 1 (Polish spaces).

Overview of Content

Lecture I: General Introduction to the Unwinding of Proofs ('Proof Mining') and first Methods of Proof Mining.

Lecture II: Logical Metatheorems 1 (Polish spaces).

Lecture III: Application to Approximation Theory. Logic Metatheorems 2 (abstract spaces). Application to Ergodic Theory

Overview of Content

Lecture I: General Introduction to the Unwinding of Proofs (‘Proof Mining’) and first Methods of Proof Mining.

Lecture II: Logical Metatheorems 1 (Polish spaces).

Lecture III: Application to Approximation Theory. Logic Metatheorems 2 (abstract spaces). Application to Ergodic Theory

Lecture IV: Applications to Fixed Point Theory and Convex Optimization.

Lecture I

Background: David Hilbert's Program

Background: David Hilbert's Program

Since the 19th century **ineffective (set-theoretic) principles** became increasingly important.

Background: David Hilbert's Program

Since the 19th century **ineffective (set-theoretic) principles** became increasingly important.

The issue of their legitimacy led Hilbert to the program:

Background: David Hilbert's Program

Since the 19th century **ineffective (set-theoretic) principles** became increasingly important.

The issue of their legitimacy led Hilbert to the program:

Establish that uses of these higher ineffective/transfinite **(,,ideal") principles** $\mathcal{I}$ in proofs of combinatorial/finitistic **(,,real") propositions** can be **eliminated**, at least in principle.

Background: David Hilbert's Program

Since the 19th century **ineffective (set-theoretic) principles** became increasingly important.

The issue of their legitimacy led Hilbert to the program:

Establish that uses of these higher ineffective/transfinite **(,,ideal") principles** $\mathcal{I}$ in proofs of combinatorial/finitistic **(,,real") propositions** can be **eliminated**, at least in principle.

In particular: Show the consistency of $\mathcal{I}$ by finitistic means.

Impossibility of the program (in the narrow sense)

Theorem [K. Gödel 1931]

For no nontrivial consistent theory $\mathcal{T}$ is it possible to prove the consistency of $\mathcal{T}$ in $\mathcal{T}$ itself.

Impossibility of the program (in the narrow sense)

Theorem [K. Gödel 1931]

For no nontrivial consistent theory $\mathcal{T}$ is it possible to prove the consistency of $\mathcal{T}$ in $\mathcal{T}$ itself.

Modified Hilbert Program:

Calibrate the contribution of the use of ideal principles in proofs.

Impossibility of the program (in the narrow sense)

Theorem [K. Gödel 1931]

For no nontrivial consistent theory $\mathcal{T}$ is it possible to prove the consistency of $\mathcal{T}$ in $\mathcal{T}$ itself.

Modified Hilbert Program:

Calibrate the contribution of the use of ideal principles in proofs.

Reduce the consistency of a theory $\mathcal{T}_1$ to that of a prima facie more constructive theory $\mathcal{T}_2$.

Impossibility of the program (in the narrow sense)

Theorem [K. Gödel 1931]

For no nontrivial consistent theory $\mathcal{T}$ is it possible to prove the consistency of $\mathcal{T}$ in $\mathcal{T}$ itself.

Modified Hilbert Program:

Calibrate the contribution of the use of ideal principles in proofs.

Reduce the consistency of a theory $\mathcal{T}_1$ to that of a prima facie more constructive theory $\mathcal{T}_2$.

In ordinary mathematics: the **"Gödel Phenomenon"** is extremely rare. Usually, "ideal" principles can be replaced by suitable more elementary ones. However: this can be very difficult to accomplish.

G. Kreisel: from consistency to mathematical applications

General malaise of consistency proofs:

G. Kreisel: from consistency to mathematical applications

General malaise of consistency proofs:

'To one who has faith, no explanation is necessary. To one without faith, no explanation is possible' (attributed to St Thomas Aquinas).

G. Kreisel: from consistency to mathematical applications

General malaise of consistency proofs:

'To one who has faith, no explanation is necessary. To one without faith, no explanation is possible' (attributed to St Thomas Aquinas).

G. Kreisel: instead of focussing on purely **universal statements** (consistency statements) consider proofs of **existential statements** which may use **arbitrary true universal axioms!**

G. Kreisel: from consistency to mathematical applications

General malaise of consistency proofs:

'To one who has faith, no explanation is necessary. To one without faith, no explanation is possible' (attributed to St Thomas Aquinas).

G. Kreisel: instead of focussing on purely **universal statements** (consistency statements) consider proofs of **existential statements** which may use **arbitrary true universal axioms!**

'What more do we know if we have proved a theorem by restricted means than if we merely know that it is true? (G. Kreisel, 50's)

Proof Mining: New results by logical analysis of proofs

Input: Noneffective proof P of C

Proof Mining: New results by logical analysis of proofs

Input: Noneffective proof P of C

Goal: Additional information on C:

Proof Mining: New results by logical analysis of proofs

Input: Noneffective proof P of C

Goal: Additional information on C:

- effective bounds,

Proof Mining: New results by logical analysis of proofs

Input: Noneffective proof P of C

Goal: Additional information on C:

- effective bounds,
- algorithms,

Proof Mining: New results by logical analysis of proofs

Input: Noneffective proof P of C

Goal: Additional information on C:

- effective bounds,
- algorithms,
- continuous dependency or full independence from certain parameters,

Proof Mining: New results by logical analysis of proofs

Input: Noneffective proof P of C

Goal: Additional information on C:

- effective bounds,
- algorithms,
- continuous dependency or full independence from certain parameters,
- generalizations of proofs: weakening of premises.

Proof Mining: New results by logical analysis of proofs

Input: Noneffective proof P of C

Goal: Additional information on C:

- effective bounds,
- algorithms,
- continuous dependency or full independence from certain parameters,
- generalizations of proofs: weakening of premises.

E.g. Let $\mathbf{C} \equiv \forall \mathbf{x} \in \mathbb{N} \exists \mathbf{y} \in \mathbb{N}\, \mathbf{F}(\mathbf{x}, \mathbf{y})$

Proof Mining: New results by logical analysis of proofs

Input: Noneffective proof P of C

Goal: Additional information on C:

- effective bounds,
- algorithms,
- continuous dependency or full independence from certain parameters,
- generalizations of proofs: weakening of premises.

E.g. Let $\mathbf{C} \equiv \forall \mathbf{x} \in \mathbb{N} \exists \mathbf{y} \in \mathbb{N}\, \mathbf{F(x, y)}$

Naive Attempt: try to extract an explicit computable function realizing (or bounding) '$\exists y$': $\forall \mathbf{x} \in \mathbb{N}\, \mathbf{F(x, f(x))}$.

Naive attempt fails

Proposition

There exist a sentence $\mathbf{A \equiv \forall x \exists y \forall z \, A_{qf}(x, y, z)}$ in the language of arithmetic (A_{qf} quantifier-free and hence decidable), such

- A is **logical valid**,

Naive attempt fails

Proposition

There exist a sentence $\mathbf{A \equiv \forall x \exists y \forall z\, A_{qf}(x, y, z)}$ in the language of arithmetic (A_{qf} quantifier-free and hence decidable), such

- A is **logical valid**,
- there is **no recursive bound f** s.t. $\mathbf{\forall x \exists y \leq f(x) \forall z\, A_{qf}(x, y, z).}$

Naive attempt fails

Proposition

There exist a sentence $\mathbf{A \equiv \forall x \exists y \forall z\, A_{qf}(x, y, z)}$ in the language of arithmetic (A_{qf} quantifier-free and hence decidable), such

- A is **logical valid**,
- there is **no recursive bound f** s.t. $\mathbf{\forall x \exists y \leq f(x) \forall z\, A_{qf}(x, y, z).}$

Proof: Take

$$\mathbf{A :\equiv \forall x \exists y \forall z \big(T(x, x, y) \vee \neg T(x, x, z)\big),}$$

where T is the (primitive recursive) Kleene-T-predicate.

Naive attempt fails

Proposition

There exist a sentence $\mathbf{A \equiv \forall x \exists y \forall z\, A_{qf}(x, y, z)}$ in the language of arithmetic (A_{qf} quantifier-free and hence decidable), such

- A is **logical valid**,
- there is **no recursive bound f** s.t. $\mathbf{\forall x \exists y \leq f(x) \forall z\, A_{qf}(x, y, z).}$

Proof: Take

$$\mathbf{A :\equiv \forall x \exists y \forall z \big(T(x, x, y) \vee \neg T(x, x, z)\big),}$$

where T is the (primitive recursive) Kleene-T-predicate.

Any bound g on '$\exists y$', i.e. no computable g such that

$$\mathbf{\forall x \exists y \leq g(x) \forall z\, (T(x, x, y) \vee \neg T(x, x, z))}$$

since this would solve the halting problem! □

However, one can obtain such **witness candidates** and bounds (and even realizing function(al)s) for a **weakened version A^H of A**:

However, one can obtain such **witness candidates** and bounds (and even realizing function(al)s) for a **weakened version** A^H **of** A:

Definition

$\mathbf{A \equiv \exists x_1 \forall y_1 \exists x_2 \forall y_2 A_{qf}(x_1, y_1, x_2, y_2)}$. Then the **Herbrand normal form** of A is defined as

$$\mathbf{A^H :\equiv \exists x_1, x_2 A_{qf}(x_1, f(x_1), x_2, g(x_1, x_2))},$$

where $\mathbf{f, g}$ are new function symbols, called index functions.

However, one can obtain such **witness candidates** and bounds (and even realizing function(al)s) for a **weakened version** A^H **of** A:

Definition

$\mathbf{A \equiv \exists x_1 \forall y_1 \exists x_2 \forall y_2 A_{qf}(x_1, y_1, x_2, y_2)}$. Then the **Herbrand normal form** of A is defined as

$$\mathbf{A^H :\equiv \exists x_1, x_2 A_{qf}(x_1, f(x_1), x_2, g(x_1, x_2))},$$

where $\mathbf{f, g}$ are new function symbols, called index functions.

A and A^H are equivalent with respect to logical validity, i.e.

$$\models \mathbf{A} \Leftrightarrow \models \mathbf{A^H},$$

but are not logically equivalent (but only in the presence of AC).

We now consider again the sentence

$$\mathbf{A \equiv \forall x\, \exists y\, \forall z\, (P(x, y) \vee \neg P(x, z))},$$

We now consider again the sentence

$$\mathbf{A \equiv \forall x\, \exists y\, \forall z\, (P(x,y) \vee \neg P(x,z)),}$$

In contrast to A, the **Herbrand normal form** A^H of A

$$\mathbf{A^H \equiv \exists y\big(P(x,y) \vee \neg P(x,g(y))\big)}$$

allows one to construct a **list of candidates** (uniformly in x, g) for '$\exists y$', namely $(c, g(c))$ for any constant c (also $(x, g(x))$)

We now consider again the sentence

$$\mathbf{A \equiv \forall x\, \exists y\, \forall z\, (P(x,y) \vee \neg P(x,z))},$$

In contrast to A, the **Herbrand normal form** A^H of A

$$\mathbf{A^H \equiv \exists y \big(P(x,y) \vee \neg P(x,g(y))\big)}$$

allows one to construct a **list of candidates** (uniformly in x, g) for '$\exists y$', namely $(c, g(c))$ for any constant c (also $(x, g(x))$)

$$\mathbf{A^{H,D} :\equiv \big(P(x,c) \vee \neg P(x,g(c))\big) \vee \big(P(x,g(c)) \vee \neg P(x,g(g(c)))\big)}$$

We now consider again the sentence

$$\mathbf{A} \equiv \forall \mathbf{x}\, \exists \mathbf{y}\, \forall \mathbf{z}\, (\mathbf{P}(\mathbf{x}, \mathbf{y}) \vee \neg \mathbf{P}(\mathbf{x}, \mathbf{z})),$$

In contrast to A, the **Herbrand normal form** A^H of A

$$\mathbf{A}^{\mathbf{H}} \equiv \exists \mathbf{y} \big(\mathbf{P}(\mathbf{x}, \mathbf{y}) \vee \neg \mathbf{P}(\mathbf{x}, \mathbf{g}(\mathbf{y}))\big)$$

allows one to construct a **list of candidates** (uniformly in x, g) for '$\exists y$', namely $(c, g(c))$ for any constant c (also $(x, g(x))$)

$$\mathbf{A}^{\mathbf{H},\mathbf{D}} :\equiv \big(\mathbf{P}(\mathbf{x}, \mathbf{c}) \vee \underbrace{\neg \mathbf{P}(\mathbf{x}, \mathbf{g}(\mathbf{c}))\big) \vee \big(\mathbf{P}(\mathbf{x}, \mathbf{g}(\mathbf{c}))}_{\in \mathrm{TAUT}} \vee \neg \mathbf{P}(\mathbf{x}, \mathbf{g}(\mathbf{g}(\mathbf{c})))\big)$$

is a tautology.

J. Herbrand's Theorem ('Théorème fondamental', 1930)

Theorem

Let $\mathbf{A} \equiv \exists \mathbf{x_1} \forall \mathbf{y_1} \exists \mathbf{x_2} \forall \mathbf{y_2} \mathbf{A_{qf}}(\mathbf{x_1}, \mathbf{y_1}, \mathbf{x_2}, \mathbf{y_2})$. Then:
PL $\vdash$ **A** iff there are terms $s_1, \ldots, s_k, t_1, \ldots, t_n$ (built up out of the constants and variables of A and the **index functions** used for the formation of A^H) such that

$$\mathbf{A^{H,D}} :\equiv \bigvee_{i=1}^{k} \bigvee_{j=1}^{n} \mathbf{A_{qf}}\big(\mathbf{s_i}, \mathbf{f}(\mathbf{s_i}), \mathbf{t_j}, \mathbf{g}(\mathbf{s_i}, \mathbf{t_j})\big)$$

is a tautology. $\mathbf{A^{H,D}}$ is called a **Herbrand Disjunction**.

J. Herbrand's Theorem ('Théorème fondamental', 1930)

Theorem

Let $\mathbf{A} \equiv \exists \mathbf{x_1} \forall \mathbf{y_1} \exists \mathbf{x_2} \forall \mathbf{y_2} \mathbf{A_{qf}}(\mathbf{x_1}, \mathbf{y_1}, \mathbf{x_2}, \mathbf{y_2})$. Then:
$\mathbf{PL} \vdash \mathbf{A}$ iff there are terms $s_1, \ldots, s_k, t_1, \ldots, t_n$ (built up out of the constants and variables of A and the **index functions** used for the formation of A^H) such that

$$\mathbf{A^{H,D}} :\equiv \bigvee_{i=1}^{k} \bigvee_{j=1}^{n} \mathbf{A_{qf}}\big(\mathbf{s_i}, \mathbf{f(s_i)}, \mathbf{t_j}, \mathbf{g(s_i, t_j)}\big)$$

is a tautology. $\mathbf{A^{H,D}}$ is called a **Herbrand Disjunction**.
Note that the length of this disjunction is fixed: $k \cdot n$.

J. Herbrand's Theorem ('Théorème fondamental', 1930)

Theorem

Let $\mathbf{A \equiv \exists x_1 \forall y_1 \exists x_2 \forall y_2 A_{qf}(x_1, y_1, x_2, y_2).}$ Then:
PL $\vdash$ **A** iff there are terms $s_1, \ldots, s_k, t_1, \ldots, t_n$ (built up out of the constants and variables of A and the **index functions** used for the formation of A^H) such that

$$\mathbf{A^{H,D} :\equiv \bigvee_{i=1}^{k} \bigvee_{j=1}^{n} A_{qf}\left(s_i, f(s_i), t_j, g(s_i, t_j)\right)}$$

is a tautology. $\mathbf{A^{H,D}}$ is called a **Herbrand Disjunction**.
Note that the length of this disjunction is fixed: $k \cdot n$. The terms s_i, t_j can be extracted from a given PL-proof of A.

Herbrand's Theorem continued

Replacing in $A^{H,D}$ all terms '$g(s_i, t_j)$', '$f(s_i)$', by new variables (treating larger terms first) results in another tautological disjunction A^{Dis} s.t. A can be inferred from A by a **direct proof**.

Remark

- For sentences $\mathbf{A} \equiv \forall x \exists y \forall z \, \mathbf{A_{qf}(x, y, z)}$, $\mathbf{A^{Dis}}$ can be written in the form

$$\mathbf{A_{qf}(x, t_1, b_1) \vee A_{qf}(x, t_2, b_2) \vee \ldots \vee A_{qf}(x, t_k, b_k),}$$

where the b_i are new variables and t_i does not contain any b_j with $i \leq j$ (used by Luckhardt's analysis of Roth's theorem, see below).

Remark

- For sentences $\mathbf{A} \equiv \forall \mathbf{x} \exists \mathbf{y} \forall \mathbf{z}\, \mathbf{A_{qf}(x, y, z)}$, $\mathbf{A^{Dis}}$ can be written in the form

$$\mathbf{A_{qf}(x, t_1, b_1) \vee A_{qf}(x, t_2, b_2) \vee \ldots \vee A_{qf}(x, t_k, b_k)},$$

  where the b_i are new variables and t_i does not contain any b_j with $i \leq j$ (used by Luckhardt's analysis of Roth's theorem, see below).
- Herbrand's theorem immediately extends to first-order theories $\mathcal{T}$ whose non-logical axioms $G_1, \ldots, G_n$ are all purely universal.

Theorem (Roth 1955)

An algebraic irrational number α has only finitely many exceptionally good rational approximations, i.e. for $\varepsilon > 0$ there are only finitely many $q \in \mathbb{N}$ such that

$$\mathbf{R(q) :\equiv q > 1 \wedge \exists! p \in \mathbb{Z} : (p, q) = 1 \wedge |\alpha - pq^{-1}| < q^{-2-\varepsilon}.}$$

Theorem (Roth 1955)

An algebraic irrational number α has only finitely many exceptionally good rational approximations, i.e. for $\varepsilon > 0$ there are only finitely many $q \in \mathbb{N}$ such that

$$\mathbf{R(q) :\equiv q > 1 \wedge \exists! p \in \mathbb{Z} : (p, q) = 1 \wedge |\alpha - pq^{-1}| < q^{-2-\varepsilon}.}$$

Theorem (Luckhardt 1985/89)

The following upper bound on $\#\{q : R(q)\}$ holds:

$$\mathbf{\#\{q : R(q)\} < \frac{7}{3}\varepsilon^{-1} \log N_\alpha + 6 \cdot 10^3 \varepsilon^{-5} \log^2 d \cdot \log(50\varepsilon^{-2} \log d),}$$

where $\mathbf{N_\alpha < \max(21 \log 2h(\alpha), 2 \log(1 + |\alpha|))}$ and h is the logarithmic absolute homogeneous height and $d = deg(\alpha)$.

Independently: Bombieri and van der Poorten 1988.

Exercise (U. Berger)

Consider open theory $\mathcal{T} := \{\forall x(S(x) \neq 0)\}$ in language with equality, constant 0 and two unary function symbols S, f.

Exercise (U. Berger)

Consider open theory $\mathcal{T} := \{\forall x(S(x) \neq 0)\}$ in language with equality, constant 0 and two unary function symbols S, f.

Proposition

$\mathcal{T} \vdash \exists \mathbf{x}\big(\mathbf{f}(\mathbf{S}(\mathbf{f}(\mathbf{x}))) \neq \mathbf{x}\big).$

Proof: Suppose that

$$\forall \mathbf{x}\big(\mathbf{f}(\mathbf{S}(\mathbf{f}(\mathbf{x}))) = \mathbf{x}\big),$$

then $\mathbf{f}$ is injective, but also (since $\mathbf{S}(\mathbf{x}) \neq \mathbf{0}$) surjective on $\{\mathbf{x} : \mathbf{x} \neq \mathbf{0}\}$ and hence non-injective. Contradiction! $\square$

Exercise (U. Berger)

Consider open theory $\mathcal{T} := \{\forall x(S(x) \neq 0)\}$ in language with equality, constant 0 and two unary function symbols S, f.

Proposition

$\mathcal{T} \vdash \exists \mathbf{x}\big(\mathbf{f(S(f(x))) \neq x}\big).$

Proof: Suppose that

$$\forall \mathbf{x}\big(\mathbf{f(S(f(x))) = x}\big),$$

then $\mathbf{f}$ is injective, but also (since $\mathbf{S(x) \neq 0}$) surjective on $\mathbf{\{x : x \neq 0\}}$ and hence non-injective. Contradiction! □

Exercise 1: Analyze the above proof to extract Herbrand terms $\mathbf{s_1, \ldots, s_k, t_1, \ldots, t_n}$ s.t.

$$\text{PL} \vdash \Big(\bigwedge_{i=1}^{k} \mathbf{S(s_i) \neq 0}\Big) \to \bigvee_{j=1}^{n} \big(\mathbf{f(S(f(t_j))) \neq t_j}\big).$$

Limitations

- Techniques work only for restricted formal contexts: mainly purely universal ('algebraic') axioms, restricted use of induction, no higher analytical principles.

Limitations

- Techniques work only for restricted formal contexts: mainly purely universal (‘algebraic’) axioms, restricted use of induction, no higher analytical principles.
- Require that one can ‘guess’ the correct Herbrand terms: in general procedure results in proofs of length $2_n^{|P|}$, where $2_{n+1}^k = 2^{2_n^k}$ (n cut complexity).

Towards generalizations of Herbrand's theorem

Allow **functionals $\Phi(x, f)$** instead of just Herbrand terms: Let's consider again the example

$$A \equiv \forall x \exists y \forall z \big(T(x, x, y) \lor \neg T(x, x, z)\big).$$

Towards generalizations of Herbrand's theorem

Allow **functionals $\Phi(x, f)$** instead of just Herbrand terms: Let's consider again the example

$$A \equiv \forall x \exists y \forall z \big(T(x,x,y) \vee \neg T(x,x,z)\big).$$

A^H can be realized by a computable functional of type level 2 which is defined by cases:

$$\Phi(x,g) := \begin{cases} c & \text{if } \neg T(x,x,g(c)) \\ g(c) & \text{otherwise.} \end{cases}$$

Towards generalizations of Herbrand's theorem

Allow **functionals $\Phi(x, f)$** instead of just Herbrand terms: Let's consider again the example

$$A \equiv \forall x \exists y \forall z \big(T(x, x, y) \vee \neg T(x, x, z)\big).$$

A^H can be realized by a computable functional of type level 2 which is defined by cases:

$$\Phi(x, g) := \begin{cases} c & \text{if } \neg T(x, x, g(c)) \\ g(c) & \text{otherwise.} \end{cases}$$

From this definition it easily follows that

$$\forall x, g \big(T(x, x, \Phi(x, g)) \vee \neg T(x, x, g(\Phi(x, g)))\big).$$

Φ satisfies **G. Kreisel's no-counterexample interpretation!**

If A is not provable in PL but e.g. in PA more **complicated functionals** are needed (Kreisel 1951):

If A is not provable in PL but e.g. in PA more **complicated functionals** are needed (Kreisel 1951):

Let (a_n) be a nonincreasing sequence in $[0,1]$. Then, clearly, (a_n) is convergent and so a Cauchy sequence which we write as:

$$\mathbf{(1)}\ \forall \mathbf{k} \in \mathbb{N} \exists \mathbf{n} \in \mathbb{N} \forall \mathbf{m} \in \mathbb{N} \forall \mathbf{i}, \mathbf{j} \in [\mathbf{n}; \mathbf{n}+\mathbf{m}]\ (|\mathbf{a_i} - \mathbf{a_j}| \leq \mathbf{2^{-k}}),$$

where $[n; n+m] := \{n, n+1, \ldots, n+m\}$.

If A is not provable in PL but e.g. in PA more **complicated functionals** are needed (Kreisel 1951):

Let (a_n) be a nonincreasing sequence in $[0,1]$. Then, clearly, (a_n) is convergent and so a Cauchy sequence which we write as:

$$\mathbf{(1)}\ \forall \mathbf{k} \in \mathbb{N} \exists \mathbf{n} \in \mathbb{N} \forall \mathbf{m} \in \mathbb{N} \forall \mathbf{i}, \mathbf{j} \in [\mathbf{n}; \mathbf{n}+\mathbf{m}]\ (|\mathbf{a_i} - \mathbf{a_j}| \leq 2^{-\mathbf{k}}),$$

where $[n; n+m] := \{n, n+1, \ldots, n+m\}$.
Then the (partial) Herbrand normal form of this statement is

$$\mathbf{(2)}\ \forall \mathbf{k} \in \mathbb{N} \forall \mathbf{g} \in \mathbb{N}^{\mathbb{N}} \exists \mathbf{n} \in \mathbb{N} \forall \mathbf{i}, \mathbf{j} \in [\mathbf{n}; \mathbf{n}+\mathbf{g}(\mathbf{n})]\ (|\mathbf{a_i} - \mathbf{a_j}| \leq 2^{-\mathbf{k}}).$$

By E. Specker 1949 there exist **computable** such sequences (a_n) even in $\mathbb{Q} \cap [0, 1]$ **without computable bound** on '$\exists n$' in (1).

By E. Specker 1949 there exist **computable** such sequences (a_n) even in $\mathbb{Q} \cap [0,1]$ **without computable bound** on '$\exists n$' in (1).

By contrast, there is a **simple (primitive recursive) bound** $\Phi^*(g,k)$ on (2) (also referred to as **'metastability'** by T.Tao):

By E. Specker 1949 there exist **computable** such sequences (a_n) even in $\mathbb{Q} \cap [0,1]$ **without computable bound** on '$\exists n$' in (1).

By contrast, there is a **simple (primitive recursive) bound** $\Phi^*(g,k)$ on (2) (also referred to as **'metastability'** by T.Tao):

Proposition

Let (a_n) be any nonincreasing sequence in $[0,1]$ then

$$\forall k \in \mathbb{N} \forall g \in \mathbb{N}^{\mathbb{N}} \exists n \leq \Phi^*(g,k) \forall i,j \in [n; n+g(n)] \ (|a_i - a_j| \leq 2^{-k}),$$

where

$$\Phi^*(g,k) := \tilde{g}^{(2^k-1)}(0) \text{ with } \tilde{g}(n) := n + g(n).$$

Moreover, there exists an $i < 2^k$ such that n can be taken as $\tilde{g}^{(i)}(0)$.

Remark

The previous result can be viewed as a polished form of a **Herbrand disjunction** of **variable (in k) length**:

$$\bigvee_{i=0}^{2^k-1} \left(|a_{\tilde{g}^{(i)}(0)} - a_{\tilde{g}(\tilde{g}^{(i)}(0))}| \le 2^{-k}\right).$$

Remark

The previous result can be viewed as a polished form of a **Herbrand disjunction** of **variable (in k) length**:

$$\bigvee_{i=0}^{2^k-1} \left(|a_{\tilde{g}^{(i)}(0)} - a_{\tilde{g}(\tilde{g}^{(i)}(0))}| \leq 2^{-k}\right).$$

Corollary (T. Tao's finite convergence principle)

$$\forall k \in \mathbb{N}, g : \mathbb{N} \to \mathbb{N} \exists M \in \mathbb{N} \forall 1 \geq a_0 \geq \ldots \geq a_M \geq 0 \exists N \in \mathbb{N} \left(N + g(N) \leq M \wedge \forall n, m \in [N, N + g(N)](|a_n - a_m| \leq 2^{-k})\right).$$

One may take $M := \tilde{g}^{(2^k)}(0)$.

No-Counterexample Interpretation (Kreisel 1951)

Recall: for a formula

$$\mathbf{A \equiv \exists x_1 \, \forall y_1 \ldots \exists x_n \, \forall y_n \, A_{qf}(x_1, y_1, \ldots, x_n, y_n)}$$

we call a tuple of functionals $\underline{\varphi}$ a **solution for the no-counterexample interpretation of** A if $\underline{\varphi}$ provides a witness for A^H

$$\mathbf{\forall f_1, \ldots, f_n \, \exists x_1, \ldots, x_n \, A_{qf}(x_1, f_1(x_1), \ldots, x_n, f_n(x_1, \ldots, x_n)),}$$

No-Counterexample Interpretation (Kreisel 1951)

Recall: for a formula

$$\mathbf{A} \equiv \exists \mathbf{x_1}\, \forall \mathbf{y_1} \ldots \exists \mathbf{x_n}\, \forall \mathbf{y_n}\, \mathbf{A_{qf}}(\mathbf{x_1}, \mathbf{y_1}, \ldots, \mathbf{x_n}, \mathbf{y_n})$$

we call a tuple of functionals $\underline{\varphi}$ a **solution for the no-counterexample interpretation of A** if $\underline{\varphi}$ provides a witness for A^H

$$\forall \mathbf{f_1}, \ldots, \mathbf{f_n}\, \exists \mathbf{x_1}, \ldots, \mathbf{x_n}\, \mathbf{A_{qf}}(\mathbf{x_1}, \mathbf{f_1}(\mathbf{x_1}), \ldots, \mathbf{x_n}, \mathbf{f_n}(\mathbf{x_1}, \ldots, \mathbf{x_n})),$$

i.e.

$$\forall \underline{\mathbf{f}}\, \mathbf{A_{qf}}(\varphi_1(\underline{\mathbf{f}}), \mathbf{f_1}(\varphi_1(\underline{\mathbf{f}})), \ldots, \varphi_n(\underline{\mathbf{f}}), \mathbf{f_n}(\varphi_1(\underline{\mathbf{f}}), \ldots, \varphi_n(\underline{\mathbf{f}}))).$$

Problems of the no-counterexample interpretation

For principles $F \in \exists\forall\exists$ n.c.i. no longer 'correct'. $C_n := \{0, 1, \ldots, n\}$.

Problems of the no-counterexample interpretation

For principles $F \in \exists\forall\exists$ n.c.i. no longer 'correct'. $C_n := \{0, 1, \ldots, n\}$.

Direct example: Infinitary Pigeonhole Principle (IPP):

$$\forall \mathbf{n} \in \mathbb{N} \forall \mathbf{f} : \mathbb{N} \to \mathbf{C_n} \exists \mathbf{i} \leq \mathbf{n} \forall \mathbf{k} \in \mathbb{N} \exists \mathbf{m} \geq \mathbf{k} \left(\mathbf{f(m) = i}\right).$$

Problems of the no-counterexample interpretation

For principles $F \in \exists\forall\exists$ n.c.i. no longer 'correct'. $C_n := \{0, 1, \ldots, n\}$.

Direct example: Infinitary Pigeonhole Principle (IPP):

$$\forall \mathbf{n} \in \mathbb{N} \forall \mathbf{f} : \mathbb{N} \to \mathbf{C_n} \exists \mathbf{i} \leq \mathbf{n} \forall \mathbf{k} \in \mathbb{N} \exists \mathbf{m} \geq \mathbf{k} \left(\mathbf{f(m) = i}\right).$$

IPP causes arbitrary **primitive recursive complexity**, but $(\mathrm{IPP})^H$

$$\forall \mathbf{n} \in \mathbb{N} \forall \mathbf{f} : \mathbb{N} \to \mathbf{C_n} \forall \mathbf{F} : \mathbf{C_n} \to \mathbb{N} \exists \mathbf{i} \leq \mathbf{n} \exists \mathbf{m} \geq \mathbf{F(i)} \left(\mathbf{f(m) = i}\right)$$

Problems of the no-counterexample interpretation

For principles $F \in \exists\forall\exists$ n.c.i. no longer 'correct'. $C_n := \{0, 1, \ldots, n\}$.

Direct example: Infinitary Pigeonhole Principle (IPP):

$$\forall \mathbf{n} \in \mathbb{N} \forall \mathbf{f} : \mathbb{N} \to \mathbf{C_n} \exists \mathbf{i} \leq \mathbf{n} \forall \mathbf{k} \in \mathbb{N} \exists \mathbf{m} \geq \mathbf{k} \left(\mathbf{f(m) = i} \right).$$

IPP causes arbitrary **primitive recursive complexity**, but $(\mathrm{IPP})^H$

$$\forall \mathbf{n} \in \mathbb{N} \forall \mathbf{f} : \mathbb{N} \to \mathbf{C_n} \forall \mathbf{F} : \mathbf{C_n} \to \mathbb{N} \exists \mathbf{i} \leq \mathbf{n} \exists \mathbf{m} \geq \mathbf{F(i)} \left(\mathbf{f(m) = i} \right)$$

has trivial n.c.i.-solution for '$\exists i$','$\exists m$':

$$\mathbf{M(n,f,F) := \max\{F(i) : i \leq n\}} \text{ and } \mathbf{I(n,f,F) := f(M(n,f,F)).}$$

Problems of the no-counterexample interpretation

For principles $F \in \exists\forall\exists$ n.c.i. no longer 'correct'. $C_n := \{0, 1, \ldots, n\}$.

Direct example: Infinitary Pigeonhole Principle (IPP):

$$\forall \mathbf{n} \in \mathbb{N} \forall \mathbf{f} : \mathbb{N} \to \mathbf{C_n} \exists \mathbf{i} \leq \mathbf{n} \forall \mathbf{k} \in \mathbb{N} \exists \mathbf{m} \geq \mathbf{k} \left(\mathbf{f(m) = i} \right).$$

IPP causes arbitrary **primitive recursive complexity**, but $(\mathrm{IPP})^H$

$$\forall \mathbf{n} \in \mathbb{N} \forall \mathbf{f} : \mathbb{N} \to \mathbf{C_n} \forall \mathbf{F} : \mathbf{C_n} \to \mathbb{N} \exists \mathbf{i} \leq \mathbf{n} \exists \mathbf{m} \geq \mathbf{F(i)} \left(\mathbf{f(m) = i} \right)$$

has trivial n.c.i.-solution for '$\exists i$','$\exists m$':

$$\mathbf{M(n, f, F) := \max\{F(i) : i \leq n\}} \text{ and } \mathbf{I(n, f, F) := f(M(n, f, F)).}$$

M, I **do not reflect** true complexity of IPP!

Problems of the no-counterexample interpretation

For principles $F \in \exists\forall\exists$ n.c.i. no longer 'correct'. $C_n := \{0, 1, \ldots, n\}$.

Direct example: Infinitary Pigeonhole Principle (IPP):

$$\forall \mathbf{n} \in \mathbb{N} \forall \mathbf{f} : \mathbb{N} \to \mathbf{C_n} \exists \mathbf{i} \leq \mathbf{n} \forall \mathbf{k} \in \mathbb{N} \exists \mathbf{m} \geq \mathbf{k} \left(\mathbf{f(m) = i}\right).$$

IPP causes arbitrary **primitive recursive complexity**, but $(\text{IPP})^H$

$$\forall \mathbf{n} \in \mathbb{N} \forall \mathbf{f} : \mathbb{N} \to \mathbf{C_n} \forall \mathbf{F} : \mathbf{C_n} \to \mathbb{N} \exists \mathbf{i} \leq \mathbf{n} \exists \mathbf{m} \geq \mathbf{F(i)} \left(\mathbf{f(m) = i}\right)$$

has trivial n.c.i.-solution for '$\exists i$','$\exists m$':

$$\mathbf{M(n, f, F) := \max\{F(i) : i \leq n\}} \text{ and } \mathbf{I(n, f, F) := f(M(n, f, F)).}$$

M, I **do not reflect** true complexity of IPP!

Related problem: bad behavior w.r.t. modus ponens!

A Modular Approach: Proof Interpretations

- **Interpret** the formulas A in $P : A \mapsto A^{\mathcal{I}}$,

A Modular Approach: Proof Interpretations

- **Interpret** the formulas A in $P : A \mapsto A^{\mathcal{I}}$,
- Interpretation $C^{\mathcal{I}}$ contains the **additional information**,

A Modular Approach: Proof Interpretations

- **Interpret** the formulas A in $P : A \mapsto A^{\mathcal{I}}$,
- Interpretation $C^{\mathcal{I}}$ contains the **additional information**,
- Construct by **recursion on** P a new proof $P^{\mathcal{I}}$ of $C^{\mathcal{I}}$.

A Modular Approach: Proof Interpretations

- **Interpret** the formulas A in $P : A \mapsto A^{\mathcal{I}}$,
- Interpretation $C^{\mathcal{I}}$ contains the **additional information**,
- Construct by **recursion on** P a new proof $P^{\mathcal{I}}$ of $C^{\mathcal{I}}$.

  In particular: solve **modus ponens problem:**

$$\frac{\mathbf{A^I} \quad , \quad \mathbf{(A \to B)^I}}{\mathbf{B^I}}.$$

A Modular Approach: Proof Interpretations

- **Interpret** the formulas A in $P : A \mapsto A^{\mathcal{I}}$,
- Interpretation $C^{\mathcal{I}}$ contains the **additional information**,
- Construct by **recursion on** P a new proof $P^{\mathcal{I}}$ of $C^{\mathcal{I}}$.

  In particular: solve **modus ponens problem:**

$$\frac{\mathbf{A^I} \quad , \quad \mathbf{(A \to B)^I}}{\mathbf{B^I}}.$$

Our approach is based on novel forms and extensions of:

K. Gödel's functional interpretation!

Detour through intuitionistic systems and higher types

Detour through intuitionistic systems and higher types

- **HA** ('Heyting arithmetic' is defined as Peano arithmetic but with **intuitionistic (constructive) logic**.

Detour through intuitionistic systems and higher types

- **HA** ('Heyting arithmetic' is defined as Peano arithmetic but with **intuitionistic (constructive) logic**.
- $\mathbf{HA}^\omega$ is the extension of **HA** to all finite types over $\mathbb{N}$.

Detour through intuitionistic systems and higher types

- **HA** ('Heyting arithmetic' is defined as Peano arithmetic but with **intuitionistic (constructive) logic**.
- **HA**$^\omega$ is the extension of **HA** to all finite types over $\mathbb{N}$.

Types T: (i) $\mathbb{N} \in \mathsf{T}$, $\quad \rho, \tau \in \mathsf{T} \Rightarrow (\rho \to \tau) \in \mathsf{T}$.

Detour through intuitionistic systems and higher types

- **HA** ('Heyting arithmetic' is defined as Peano arithmetic but with **intuitionistic (constructive) logic**.
- $\mathbf{HA}^\omega$ is the extension of **HA** to all finite types over $\mathbb{N}$.

Types T: (i) $\mathbb{N} \in \mathsf{T}$, $\quad \rho, \tau \in \mathsf{T} \Rightarrow (\rho \to \tau) \in \mathsf{T}$.

$\mathbf{HA}^\omega$ has λ-abstraction $(\lambda x^\rho . t[x]^\tau)(s^\rho) =_\tau t[s/x]$ and **primitive recursion in all finite types** (Hilbert 1926, Gödel 1958): for $x \in \mathbb{N}$

$$\mathbf{R}_\rho(\mathbf{0}, \mathbf{y}, \mathbf{z}) =_\rho \mathbf{y}, \;\; \mathbf{R}_\rho(\mathbf{x}+\mathbf{1}, \mathbf{y}, \mathbf{z}) =_\rho \mathbf{z}(\mathbf{R}_\rho \mathbf{xyz}, \mathbf{x}),$$

where $=_\rho$ is defined as pointwise (extensional) equality.

Detour through intuitionistic systems and higher types

- **HA** ('Heyting arithmetic' is defined as Peano arithmetic but with **intuitionistic (constructive) logic**.
- **HA**$^\omega$ is the extension of **HA** to all finite types over $\mathbb{N}$.

Types T: (i) $\mathbb{N} \in \mathrm{T}$, $\rho, \tau \in \mathrm{T} \Rightarrow (\rho \to \tau) \in \mathrm{T}$.

HA$^\omega$ has λ-abstraction $(\lambda x^\rho . t[x]^\tau)(s^\rho) =_\tau t[s/x]$ and **primitive recursion in all finite types** (Hilbert 1926, Gödel 1958): for $x \in \mathbb{N}$

$$\mathbf{R}_\rho(0, y, z) =_\rho y, \ \mathbf{R}_\rho(x+1, y, z) =_\rho z(\mathbf{R}_\rho xyz, x),$$

where $=_\rho$ is defined as pointwise (extensional) equality.

PA$^\omega$ =**HA**$^\omega + (A \vee \neg A)$.

Exercise: Show that primitive recursion in higher types defines more functions $f : \mathbb{N} \to \mathbb{N}$ than the usual primitive recursive ones, e.g. it defines the Ackermann function $Ack(x) := A(x, x)$, where

$$\begin{cases} \mathbf{A(0, y) := y + 1,} \\ \mathbf{A(x + 1, 0) := A(x, 1),} \\ \mathbf{A(x + 1, y + 1) := A(x, A(x + 1, y)).} \end{cases}$$

ω-Models of PA^ω

ω-Models of PA^ω

- Full set-theoretic type structure $\mathcal{S}^\omega := \langle \mathsf{S}_\rho \rangle_{\rho \in \mathrm{T}}$:

$$\mathsf{S}_{\rho\to\tau} := \{ \text{ all set-theoretic functions: } \mathsf{S}_\rho \to \mathsf{S}_\tau \}.$$

ω-Models of PA^ω

- Full set-theoretic type structure $\mathcal{S}^\omega := \langle \mathbf{S}_\rho \rangle_{\rho \in \mathrm{T}}$:

$$\mathbf{S}_{\rho\to\tau} := \{ \text{ all set-theoretic functions: } \mathbf{S}_\rho \to \mathbf{S}_\tau \}.$$

- Continuous functionals $\mathcal{C}^\omega := \langle \mathbf{C}_\rho \rangle_{\rho \in \mathrm{T}}$:

$$\mathbf{C}_{\rho\to\tau} := \{ \text{ all sequentially continuous (Kuratowski) functions: } \mathbf{C}_\rho \to \mathbf{C}_\tau \}.$$

ω-Models of PA^ω

- Full set-theoretic type structure $\boldsymbol{\mathcal{S}^\omega := \langle \mathbf{S}_\rho \rangle_{\rho \in \mathrm{T}}}$:

  $$\mathbf{S}_{\rho\to\tau} := \{ \text{ all set-theoretic functions: } \mathbf{S}_\rho \to \mathbf{S}_\tau \}.$$

- Continuous functionals $\boldsymbol{\mathcal{C}^\omega := \langle \mathbf{C}_\rho \rangle_{\rho \in \mathrm{T}}}$:

  $\mathbf{C}_{\rho\to\tau} := \{$ all sequentially continuous (Kuratowski) functions: $\mathbf{C}_\rho \to \mathbf{C}_\tau \}.$

- Majorizable functionals (see below) $\boldsymbol{\mathcal{M}^\omega := \langle \mathbf{M}_\rho \rangle_{\rho \in \mathrm{T}}}$:

  $\mathbf{M}_{\rho\to\tau} := \{$ all majorizable (Howard-Bezem) functions : $\mathbf{M}_\rho \to \mathbf{M}_\tau \}.$

Further exercises

- Prove that $\forall n, m \in \mathbb{N}^* \, (\sqrt{2} \neq \frac{n}{m})$ and extract from the proof an effective irrationality measure $f : \mathbb{N}^* \to \mathbb{N}^*$, i.e.

$$\forall \mathbf{n}, \mathbf{m} \in \mathbb{N}^* \, (|\sqrt{\mathbf{2}} - \mathbf{n}/\mathbf{m}| \geq \mathbf{1}/\mathbf{f}(\mathbf{m})).$$

Further exercises

- Prove that $\forall n, m \in \mathbb{N}^* \, (\sqrt{2} \neq \frac{n}{m})$ and extract from the proof an effective irrationality measure $f : \mathbb{N}^* \to \mathbb{N}^*$, i.e.

$$\forall \mathbf{n}, \mathbf{m} \in \mathbb{N}^* \, (|\sqrt{\mathbf{2}} - \mathbf{n}/\mathbf{m}| \geq \mathbf{1}/\mathbf{f}(\mathbf{m})).$$

- Prove that

$$\forall \mathbf{f} \in \mathbb{N}^{\mathbb{N}} \, \forall \mathbf{k} \in \mathbb{N} \, \exists \mathbf{n} \geq \mathbf{k} \, (\mathbf{f}(\mathbf{n}) \leq \min\{\mathbf{f}(\mathbf{3n}), \mathbf{f}(\mathbf{n}^2)\})$$

and extract a (prim. rec.) bound $\Phi(f, k)$ such that

$$\forall \mathbf{f} \in \mathbb{N}^{\mathbb{N}} \, \forall \mathbf{k} \in \mathbb{N} \exists \mathbf{n} \leq \mathbf{\Phi}(\mathbf{f}, \mathbf{k}) \, (\mathbf{n} \geq \mathbf{k} \wedge \mathbf{f}(\mathbf{n}) \leq \min\{\mathbf{f}(\mathbf{3n}), \mathbf{f}(\mathbf{n}^2))\}.$$

- Let $(a_n), (b_n), (c_n)$ be sequences in $\mathbb{R}_+$ s.t. $\sum a_n, \sum b_n < \infty$ and

$$\forall n \in \mathbb{N}\, (a_{n+1} \leq (1+b_n)a_n + c_n).$$

Construct a primitive recursive functional $\Phi(g,k) = \Phi(A,B,C,g,k)$ s.t.

$$\forall k \in \mathbb{N}\, \forall g \in \mathbb{N}^{\mathbb{N}}\, \exists n \leq \Phi(g,k)\, \forall i,j \in [n; n+g(n)]\, (|a_i - a_j| < 2^{-k}),$$

where $a_0 \leq A, \sum b_n \leq B, \sum c_n \leq C$.

Literature

1) Gerhardy, P., Kohlenbach, U., Extracting Herbrand disjunctions by functional interpretation. Arch. Math. Logic. vol. 44, pp. 633-644 (2005).

2) Girard, J.-Y., Proof Theory and Logical Complexity Vol.I.

Studies in Proof Theory. Bibliopolis (Napoli) and Elsevier Science Publishers (Amsterdam) 1987.

3) Kohlenbach, U., Applied Proof Theory: Proof Interpretations and their Use in Mathematics. Springer Monographs in Mathematics. xx+536pp., Springer Heidelberg-Berlin, 2008.

4) Kreisel, G., Finiteness theorems in arithmetic: an application of Herbrand's theorem for Σ_2-formulas. Proc. of the Herbrand symposium (Marseille, 1981), North-Holland (Amsterdam), pp. 39-55 (1982).

5) Kreisel, G., On the interpretation of non-finitist proofs, part I. J. Symbolic Logic **16**, pp.241-267 (1951).

6) Kreisel, G., On the interpretation of non-finitist proofs, part II: Interpretation of number theory, applications. J. Symbolic Logic **17**, pp. 43-58 (1952).

7) Kreisel, G., Macintyre, A., Constructive logic versus algebraization I. In: Troelstra, A.S., van Dalen, D. (eds.), Proc. L.E.J. Brouwer Centenary Symposium (Noordwijkerhout 1981), North-Holland (Amsterdam), pp. 217-260 (1982).

8) Luckhardt, H., Herbrand-Analysen zweier Beweise des Satzes von Roth: Polynomiale Anzahlschranken. J. Symbolic Logic **54**, pp. 234-263 (1989).

9) Macintyre, A., The mathematical significance of proof theory. Phil. Trans. R. Soc. A **363**, pp. 2419-2435 (2005).

10) Tao, T., Soft analysis, hard analysis, and the finite convergence principle. In: 'Structure and Randomness. AMS, 298pp., 2008'.

11) Troelstra, A.S. (ed.) Metamathematical investigation of intuitionistic arithmetic and analysis. Springer LNM **344** (1973).

Lecture II

Detour through intuitionistic systems and higher types

Detour through intuitionistic systems and higher types

- **HA** ('Heyting arithmetic' is defined as Peano arithmetic but with **intuitionistic (constructive) logic**.

Detour through intuitionistic systems and higher types

- **HA** ('Heyting arithmetic' is defined as Peano arithmetic but with **intuitionistic (constructive) logic**.
- **HA**$^\omega$ is the extension of **HA** to all finite types over $\mathbb{N}$.

Detour through intuitionistic systems and higher types

- **HA** ('Heyting arithmetic' is defined as Peano arithmetic but with **intuitionistic (constructive) logic**.
- $\mathbf{HA}^\omega$ is the extension of **HA** to all finite types over $\mathbb{N}$.

Types T: (i) $\mathbb{N} \in \mathsf{T}, \quad \rho, \tau \in \mathsf{T} \Rightarrow (\rho \to \tau) \in \mathsf{T}$.

Detour through intuitionistic systems and higher types

- **HA** ('Heyting arithmetic' is defined as Peano arithmetic but with **intuitionistic (constructive) logic**.
- **HA**$^\omega$ is the extension of **HA** to all finite types over $\mathbb{N}$.

Types T: (i) $\mathbb{N} \in \mathsf{T}$, $\quad \rho, \tau \in \mathsf{T} \Rightarrow (\rho \to \tau) \in \mathsf{T}$.

HA$^\omega$ has λ-abstraction $(\lambda x^\rho.t[x]^\tau)(s^\rho) =_\tau t[s/x]$ and **primitive recursion in all finite types** (Hilbert 1926, Gödel 1958): for $x \in \mathbb{N}$

$$\mathbf{R}_\rho(\mathbf{0}, \mathbf{y}, \mathbf{z}) =_\rho \mathbf{y}, \ \mathbf{R}_\rho(\mathbf{x}+\mathbf{1}, \mathbf{y}, \mathbf{z}) =_\rho \mathbf{z}(\mathbf{R}_\rho \mathbf{xyz}, \mathbf{x}),$$

where $=_\rho$ is defined as pointwise (extensional) equality.

Detour through intuitionistic systems and higher types

- **HA** ('Heyting arithmetic' is defined as Peano arithmetic but with **intuitionistic (constructive) logic**.
- $\mathbf{HA}^\omega$ is the extension of **HA** to all finite types over $\mathbb{N}$.

Types T: (i) $\mathbb{N} \in \mathrm{T}$, $\rho, \tau \in \mathrm{T} \Rightarrow (\rho \to \tau) \in \mathrm{T}$.

$\mathbf{HA}^\omega$ has λ-abstraction $(\lambda x^\rho . t[x]^\tau)(s^\rho) =_\tau t[s/x]$ and **primitive recursion in all finite types** (Hilbert 1926, Gödel 1958): for $x \in \mathbb{N}$

$$\mathbf{R}_\rho(\mathbf{0}, \mathbf{y}, \mathbf{z}) =_\rho \mathbf{y}, \ \mathbf{R}_\rho(\mathbf{x}+\mathbf{1}, \mathbf{y}, \mathbf{z}) =_\rho \mathbf{z}(\mathbf{R}_\rho \mathbf{xyz}, \mathbf{x}),$$

where $=_\rho$ is defined as pointwise (extensional) equality.

$\mathbf{PA}^\omega = \mathbf{HA}^\omega + (A \vee \neg A)$.


In 2007 is het 100 jaar geleden dat L.E.J. Brouwer (1881 – 1966) de stelling van Aristoteles verwierp. Brouwer vond dat een wiskundige stelling pas waar is als er 'Positief Bewijs' is. Brouwer is de grondlegger van de intuïtionistische wiskunde.
Naar hem is o.a. de dekpuntstelling van Brouwer vernoemd. Iedere drie jaar reikt het Koninklijk Wiskundig Genootschap de Brouwer medaille uit aan een belangrijk wiskundige. Voor meer informatie: www.knaw.nl
NEDERLAND
€0,44
$\nvDash A \vee \neg A$
Er is positief bewijs!
100 jaar na dato wordt de wiskundige
L.E.J. Brouwer (1881-1966)
geëerd met een eigen postzegel.

Proof mining in the intuitionistic case: modified realizability (Kreisel 1959)

To each $A \in \mathcal{L}(\mathrm{HA}^\omega)$ we assign a new formula $\underline{x}$ *mr* A ('$\underline{x}$ modified realizes A') inductively by

Proof mining in the intuitionistic case: modified realizability (Kreisel 1959)

To each $A \in \mathcal{L}(\mathrm{HA}^\omega)$ we assign a new formula $\underline{x}\ mr\ A$ ('$\underline{x}$ modified realizes A') inductively by

(i) $\underline{x}\ mr\ A :\equiv A$ with the empty tuple $\underline{x}$, if A is a prime formula.

(ii) $\underline{x}, \underline{y}\ mr\ (A \wedge B) :\equiv \underline{x}\ mr\ A \wedge \underline{y}\ mr\ B$.

(iii) $z^{\mathbb{N}}, \underline{x}, \underline{y}\ mr\ (A \vee B) :\equiv [(z = 0 \to \underline{x}\ mr\ A) \wedge (z \neq 0 \to \underline{y}\ mr\ B)]$.

(iv) $\underline{y}\ mr\ (A \to B) :\equiv \forall \underline{x}(\underline{x}\ mr\ A \to \underline{y}\,\underline{x}\ mr\ B)$.

(v) $\underline{x}\ mr\ (\forall y^\rho A(y)) :\equiv \forall y^\rho(\underline{x}y\ mr\ A(y))$.

(vi) $z^\rho, \underline{x}\ mr\ (\exists y^\rho A(y)) :\equiv \underline{x}\ mr\ A(z)$.

Program extraction by modified realizability

Axiom of choice (in all types)

$$\mathrm{AC}: \quad \forall \mathbf{a}^{\alpha}\, \exists \mathbf{b}^{\beta}\, \mathbf{F}(\mathbf{a},\mathbf{b}) \rightarrow \exists \mathbf{B}^{\rho\rightarrow\tau}\, \forall \mathbf{a}^{\rho}\, \mathbf{F}(\mathbf{a},\mathbf{B}(\mathbf{a})).$$

Theorem

From a proof of

$$\mathrm{HA}^{\omega}+\mathrm{AC} \vdash \forall \mathbf{x}^{\rho}\left(\neg \mathbf{B}(\mathbf{x}) \rightarrow \exists \mathbf{y}^{\tau}\, \mathbf{A}(\mathbf{x},\mathbf{y})\right)$$

one can extract by *mr* a primitive recursive functional Φ s.t.

$$\mathcal{S}^{\omega} \models \forall \mathbf{x}^{\rho}\left(\neg \mathbf{B}(\mathbf{x}) \rightarrow \mathbf{A}(\mathbf{x},\Phi(\mathbf{x}))\right)$$

(A, B, ρ, τ arbitrary).

Towards proofs based on classical logic

Problem: Cannot be used with classical logic as negative translation (very roughly: $\exists \mapsto \neg\neg\exists$) of $\mathbf{PA}^\omega$ into $\mathbf{HA}^\omega$ always results in empty realizers!

Towards proofs based on classical logic

Problem: Cannot be used with classical logic as negative translation (very roughly: $\exists \mapsto \neg\neg\exists$) of $\mathbf{PA}^\omega$ into $\mathbf{HA}^\omega$ always results in empty realizers!

Entrance door for classical logic: Markov's principle $\mathbf{M}^\omega$!

$$\mathbf{M}^\omega: \neg\neg\exists \mathbf{x}^\rho\, \mathbf{A}_{\mathbf{qf}}(\mathbf{x}) \rightarrow \exists \mathbf{x}^\rho\, \mathbf{A}_{\mathbf{qf}}(\mathbf{x}), \quad \mathbf{A}_{\mathbf{qf}} \text{ quantifier-free.}$$

Towards proofs based on classical logic

Problem: Cannot be used with classical logic as negative translation (very roughly: $\exists \mapsto \neg\neg\exists$) of $\mathbf{PA}^\omega$ into $\mathbf{HA}^\omega$ always results in empty realizers!

Entrance door for classical logic: Markov's principle $\mathbf{M}^\omega$!

$$\mathbf{M}^{\omega} : \neg\neg\exists \mathbf{x}^{\rho}\, \mathbf{A}_{\mathrm{qf}}(\mathbf{x}) \rightarrow \exists \mathbf{x}^{\rho}\, \mathbf{A}_{\mathrm{qf}}(\mathbf{x}), \quad \mathbf{A}_{\mathrm{qf}} \text{ quantifier-free.}$$

For $\rho = \mathbb{N}$, this has a partial computable solution by unbounded search (no complexity information), but no total computable solution of *mr*!

Towards proofs based on classical logic

Problem: Cannot be used with classical logic as negative translation (very roughly: $\exists \mapsto \neg\neg\exists$) of $\mathbf{PA}^\omega$ into $\mathbf{HA}^\omega$ always results in empty realizers!

Entrance door for classical logic: Markov's principle $\mathbf{M}^\omega$!

$$\mathbf{M}^\omega : \neg\neg\exists x^\rho A_{qf}(x) \rightarrow \exists x^\rho A_{qf}(x), \quad A_{qf} \text{ quantifier-free.}$$

For $\rho = \mathbb{N}$, this has a partial computable solution by unbounded search (no complexity information), but no total computable solution of *mr*!

For $\rho \neq \mathbb{N}$: not even unbounded search possible!

Gödel's functional ('Dialectica') interpretation D (Gödel 1941, 1958)

Solution: Don't try to solve $\mathbf{M}^{\omega}$ but eliminate it from proofs!

Gödel's functional ('Dialectica') interpretation D (Gödel 1941, 1958)

Solution: Don't try to solve $\mathbf{M}^\omega$ but eliminate it from proofs!

Gödel's D assigns to each A a formula $A^D \equiv \exists \underline{x}\, \forall \underline{y}\, A_D(\underline{x}, \underline{y})$ (A_D qf).

Gödel's functional ('Dialectica') interpretation D (Gödel 1941, 1958)

Solution: Don't try to solve $\mathbf{M}^\omega$ but eliminate it from proofs!

Gödel's D assigns to each A a formula $A^D \equiv \exists \underline{x}\, \forall \underline{y}\, A_D(\underline{x}, \underline{y})$ (A_D qf).

Interpretation differs from *mr* for the clause of '$\to$'

$$\mathbf{(A \to B)^D :\equiv \exists \underline{U}\, \underline{Y}\, \forall \underline{x}\, \underline{v}\, \big(\underbrace{A_D(\underline{x}, \underline{Y}\underline{x}\,\underline{v}) \to B_D(\underline{U}\underline{x}, \underline{v})}_{(A \to B)_D :\equiv} \big).}$$

Gödel's functional ('Dialectica') interpretation D (Gödel 1941, 1958)

Solution: Don't try to solve $\mathbf{M}^\omega$ but eliminate it from proofs!

Gödel's D assigns to each A a formula $A^D \equiv \exists \underline{x} \forall \underline{y} A_D(\underline{x}, \underline{y})$ (A_D qf).

Interpretation differs from *mr* for the clause of '$\to$'

$$(\mathbf{A} \to \mathbf{B})^{\mathbf{D}} :\equiv \exists \underline{\mathbf{U}}\, \underline{\mathbf{Y}}\, \forall \underline{\mathbf{x}}\, \underline{\mathbf{v}} \Big(\underbrace{\mathbf{A_D}(\underline{\mathbf{x}}, \underline{\mathbf{Y}\mathbf{x}}\, \underline{\mathbf{v}}) \to \mathbf{B_D}(\underline{\mathbf{U}\mathbf{x}}, \underline{\mathbf{v}})}_{(\mathbf{A}\to\mathbf{B})_{\mathbf{D}}:\equiv} \Big).$$

Then

$$\mathbf{M^D} \equiv \exists \mathbf{x}\, \neg\neg \mathbf{A_{qf}}(\mathbf{x}) \to \exists \mathbf{x}\, \mathbf{A_{qf}}(\mathbf{x}) \quad (\text{trivial since } \mathbf{A_{qf}} \text{ decidable}).$$

Gödel's functional ('Dialectica') interpretation D (Gödel 1941, 1958)

Solution: Don't try to solve $\mathbf{M}^\omega$ but eliminate it from proofs!

Gödel's D assigns to each A a formula $A^D \equiv \exists \underline{x} \forall \underline{y} A_D(\underline{x}, \underline{y})$ (A_D qf).

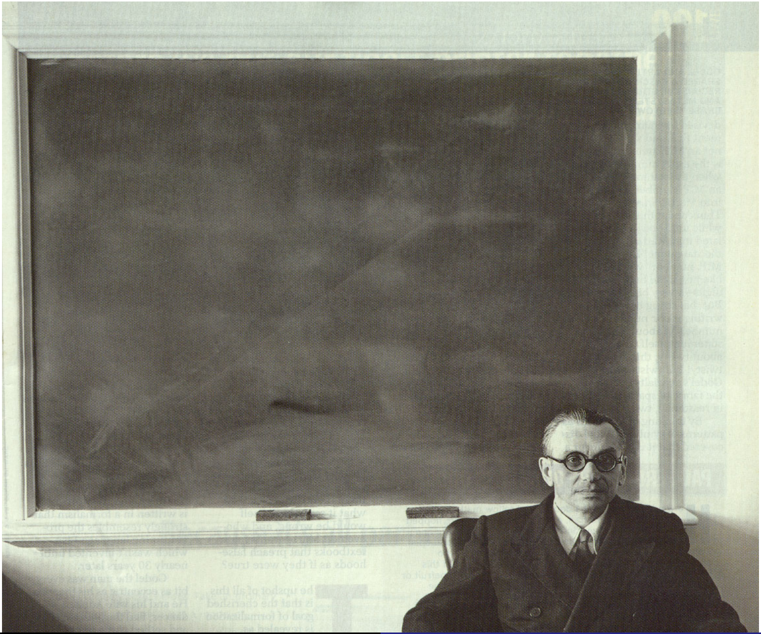
Interpretation differs from *mr* for the clause of '$\to$'

$$\mathbf{(A \to B)^D :\equiv \exists \underline{U}\, \underline{Y}\, \forall \underline{x}\, \underline{v} \big(\underbrace{A_D(\underline{x}, \underline{Y}\underline{x}\, \underline{v}) \to B_D(\underline{U}\underline{x}, \underline{v})}_{(A\to B)_D :\equiv} \big).}$$

Then

$$\mathbf{M^D \equiv \exists x \neg\neg A_{qf}(x) \to \exists x\, A_{qf}(x)} \quad \textbf{(}\text{trivial since } \mathbf{A_{qf}} \text{ decidable}\textbf{).}$$

Partial alternative: Friedman-Dragalin A-interpretation after negative translation as intermediate step (variants: Schwichtenberg, Coquand-Hofmann).



Program extraction by D

Theorem

From a proof of

$$\mathbf{HA^\omega + AC + M^\omega} \ \vdash \forall \mathbf{x}^\rho \left(\forall \mathbf{u}\, \mathbf{B_{qf}(x, u)} \rightarrow \exists \mathbf{y}^\tau\, \mathbf{A(x, y)}\right)$$

one can extract by D a primitive recursive functionals Φ s.t.

$$\mathcal{S}^\omega \models \forall \mathbf{x}^\rho \left(\forall \mathbf{u}\, \mathbf{B_{qf}(x, u)} \rightarrow \mathbf{A(x, \Phi(x))}\right)$$

$(A, \rho, \tau$ arbitrary, B_{qf} quantifier-free).

Gödel's functional interpretation in five minutes

Gödel's **functional interpretation** D combined with Krivine's **negative translation** N results in an interpretation $Sh = D \circ N$ (Streicher/K.07)

$$\mathbf{A} \mapsto \mathbf{A}^{\mathbf{Sh}} \quad \text{(Shoenfield variant)}$$

such that

Gödel's functional interpretation in five minutes

Gödel's **functional interpretation** D combined with Krivine's **negative translation** N results in an interpretation $Sh = D \circ N$ (Streicher/K.07)

$$\mathbf{A} \mapsto \mathbf{A}^{\mathbf{Sh}} \quad \text{(Shoenfield variant)}$$

such that

- $\mathbf{A}^{\mathbf{Sh}} \equiv \forall \underline{\mathbf{x}} \exists \underline{\mathbf{y}}\, \mathbf{A}_{\mathbf{Sh}}(\underline{\mathbf{x}}, \underline{\mathbf{y}})$, where $\mathbf{A}_{\mathbf{Sh}}$ is **quantifier-free**,

Gödel's functional interpretation in five minutes

Gödel's **functional interpretation** D combined with Krivine's **negative translation** N results in an interpretation $Sh = D \circ N$ (Streicher/K.07)

$$\mathbf{A} \mapsto \mathbf{A^{Sh}} \quad \text{(Shoenfield variant)}$$

such that

- $\mathbf{A^{Sh}} \equiv \forall \underline{\mathbf{x}} \exists \underline{\mathbf{y}}\, \mathbf{A_{Sh}(\underline{x}, \underline{y})}$, where $\mathbf{A_{Sh}}$ is **quantifier-free**,
- For $\mathbf{A} \equiv \forall \underline{\mathbf{x}} \exists \underline{\mathbf{y}}\, \mathbf{A_{qf}(\underline{x}, \underline{y})}$ one has $\mathbf{A^{Sh}} \equiv \mathbf{A}$.

Gödel's functional interpretation in five minutes

Gödel's **functional interpretation** D combined with Krivine's **negative translation** N results in an interpretation $Sh = D \circ N$ (Streicher/K.07)

$$\mathbf{A} \mapsto \mathbf{A^{Sh}} \quad \text{(Shoenfield variant)}$$

such that

- $\mathbf{A^{Sh}} \equiv \forall \underline{\mathbf{x}} \exists \underline{\mathbf{y}}\, \mathbf{A_{Sh}}(\underline{\mathbf{x}}, \underline{\mathbf{y}})$, where $\mathbf{A_{Sh}}$ is **quantifier-free**,
- For $\mathbf{A} \equiv \forall \underline{\mathbf{x}} \exists \underline{\mathbf{y}}\, \mathbf{A_{qf}}(\underline{\mathbf{x}}, \underline{\mathbf{y}})$ one has $\mathbf{A^{Sh}} \equiv \mathbf{A}$.
- $\underline{\mathbf{x}}, \underline{\mathbf{y}}$ are tuples of **functionals of finite type** over the base types of the system at hand.

$$A^{Sh} \equiv \forall u \exists x \, A_{Sh}(u, x), \quad B^{Sh} \equiv \forall v \exists y \, B_{Sh}(v, y).$$

$$\mathbf{A}^{\mathbf{Sh}} \equiv \forall \mathbf{u} \exists \mathbf{x}\, \mathbf{A}_{\mathbf{Sh}}(\mathbf{u}, \mathbf{x}), \quad \mathbf{B}^{\mathbf{Sh}} \equiv \forall \mathbf{v} \exists \mathbf{y}\, \mathbf{B}_{\mathbf{Sh}}(\mathbf{v}, \mathbf{y}).$$

(Sh1) $\mathbf{P}^{\mathbf{Sh}} \equiv \mathbf{P} \equiv \mathbf{P}_{\mathbf{Sh}}$ for atomic $\mathbf{P}$

$$A^{Sh} \equiv \forall u \exists x\, A_{Sh}(u, x), \quad B^{Sh} \equiv \forall v \exists y\, B_{Sh}(v, y).$$

(Sh1) $P^{Sh} \equiv P \equiv P_{Sh}$ for atomic P

(Sh2) $(\neg A)^{Sh} \equiv \forall f \exists u\, \neg A_{Sh}(u, f(u))$

$$\mathbf{A}^{\mathbf{Sh}} \equiv \forall \mathbf{u} \exists \mathbf{x}\, \mathbf{A}_{\mathbf{Sh}}(\mathbf{u}, \mathbf{x}), \quad \mathbf{B}^{\mathbf{Sh}} \equiv \forall \mathbf{v} \exists \mathbf{y}\, \mathbf{B}_{\mathbf{Sh}}(\mathbf{v}, \mathbf{y}).$$

(Sh1) $\mathbf{P}^{\mathbf{Sh}} \equiv \mathbf{P} \equiv \mathbf{P}_{\mathbf{Sh}}$ for atomic $\mathbf{P}$

(Sh2) $(\neg \mathbf{A})^{\mathbf{Sh}} \equiv \forall \mathbf{f} \exists \mathbf{u}\, \neg \mathbf{A}_{\mathbf{Sh}}(\mathbf{u}, \mathbf{f}(\mathbf{u}))$

(Sh3) $(\mathbf{A} \vee \mathbf{B})^{\mathbf{Sh}} \equiv \forall \mathbf{u}, \mathbf{v} \exists \mathbf{x}, \mathbf{y} \left(\mathbf{A}_{\mathbf{Sh}}(\mathbf{u}, \mathbf{x}) \vee \mathbf{B}_{\mathbf{Sh}}(\mathbf{v}, \mathbf{y})\right)$

$$A^{Sh} \equiv \forall u \exists x\, A_{Sh}(u, x), \quad B^{Sh} \equiv \forall v \exists y\, B_{Sh}(v, y).$$

(Sh1) $P^{Sh} \equiv P \equiv P_{Sh}$ for atomic P

(Sh2) $(\neg A)^{Sh} \equiv \forall f \exists u\, \neg A_{Sh}(u, f(u))$

(Sh3) $(A \vee B)^{Sh} \equiv \forall u, v \exists x, y\, \big(A_{Sh}(u, x) \vee B_{Sh}(v, y)\big)$

(Sh4) $(\forall z\, A)^{Sh} \equiv \forall z, u \exists x\, A_{Sh}(z, u, x)$

$$A^{Sh} \equiv \forall u \exists x \, A_{Sh}(u,x), \quad B^{Sh} \equiv \forall v \exists y \, B_{Sh}(v,y).$$

(Sh1) $P^{Sh} \equiv P \equiv P_{Sh}$ for atomic P

(Sh2) $(\neg A)^{Sh} \equiv \forall f \exists u \, \neg A_{Sh}(u, f(u))$

(Sh3) $(A \vee B)^{Sh} \equiv \forall u, v \exists x, y \left(A_{Sh}(u,x) \vee B_{Sh}(v,y)\right)$

(Sh4) $(\forall z \, A)^{Sh} \equiv \forall z, u \exists x \, A_{Sh}(z,u,x)$

(Sh5) $(A \rightarrow B)^{Sh} \equiv \forall f, v \exists u, y \left(A_{Sh}(u, f(u)) \rightarrow B_{Sh}(v,y)\right)$

$$\mathbf{A}^{\mathbf{Sh}} \equiv \forall \mathbf{u} \exists \mathbf{x}\, \mathbf{A}_{\mathbf{Sh}}(\mathbf{u}, \mathbf{x}), \quad \mathbf{B}^{\mathbf{Sh}} \equiv \forall \mathbf{v} \exists \mathbf{y}\, \mathbf{B}_{\mathbf{Sh}}(\mathbf{v}, \mathbf{y}).$$

(Sh1) $\mathbf{P}^{\mathbf{Sh}} \equiv \mathbf{P} \equiv \mathbf{P}_{\mathbf{Sh}}$ for atomic $\mathbf{P}$

(Sh2) $(\neg \mathbf{A})^{\mathbf{Sh}} \equiv \forall \mathbf{f} \exists \mathbf{u}\, \neg \mathbf{A}_{\mathbf{Sh}}(\mathbf{u}, \mathbf{f}(\mathbf{u}))$

(Sh3) $(\mathbf{A} \vee \mathbf{B})^{\mathbf{Sh}} \equiv \forall \mathbf{u}, \mathbf{v} \exists \mathbf{x}, \mathbf{y}\, \big(\mathbf{A}_{\mathbf{Sh}}(\mathbf{u}, \mathbf{x}) \vee \mathbf{B}_{\mathbf{Sh}}(\mathbf{v}, \mathbf{y})\big)$

(Sh4) $(\forall \mathbf{z}\, \mathbf{A})^{\mathbf{Sh}} \equiv \forall \mathbf{z}, \mathbf{u} \exists \mathbf{x}\, \mathbf{A}_{\mathbf{Sh}}(\mathbf{z}, \mathbf{u}, \mathbf{x})$

(Sh5) $(\mathbf{A} \rightarrow \mathbf{B})^{\mathbf{Sh}} \equiv \forall \mathbf{f}, \mathbf{v} \exists \mathbf{u}, \mathbf{y}\, \big(\mathbf{A}_{\mathbf{Sh}}(\mathbf{u}, \mathbf{f}(\mathbf{u})) \rightarrow \mathbf{B}_{\mathbf{Sh}}(\mathbf{v}, \mathbf{y})\big)$

(Sh6) $(\exists \mathbf{z} \mathbf{A})^{\mathbf{Sh}} \equiv \forall \mathbf{U} \exists \mathbf{z}, \mathbf{f}\, \mathbf{A}_{\mathbf{Sh}}(\mathbf{z}, \mathbf{U}(\mathbf{z}, \mathbf{f}), \mathbf{f}(\mathbf{U}(\mathbf{z}, \mathbf{f})))$

$$A^{Sh} \equiv \forall u \exists x\, A_{Sh}(u,x), \quad B^{Sh} \equiv \forall v \exists y\, B_{Sh}(v,y).$$

(Sh1) $P^{Sh} \equiv P \equiv P_{Sh}$ for atomic P

(Sh2) $(\neg A)^{Sh} \equiv \forall f \exists u\, \neg A_{Sh}(u, f(u))$

(Sh3) $(A \vee B)^{Sh} \equiv \forall u, v \exists x, y\, \big(A_{Sh}(u,x) \vee B_{Sh}(v,y)\big)$

(Sh4) $(\forall z\, A)^{Sh} \equiv \forall z, u \exists x\, A_{Sh}(z,u,x)$

(Sh5) $(A \to B)^{Sh} \equiv \forall f, v \exists u, y\, \big(A_{Sh}(u, f(u)) \to B_{Sh}(v,y)\big)$

(Sh6) $(\exists z A)^{Sh} \equiv \forall U \exists z, f\, A_{Sh}(z, U(z,f), f(U(z,f)))$

(Sh7) $(A \wedge B)^{Sh} \equiv$
$\forall n, u, v \exists x, y\, (n{=}0 \to A_{Sh}(u,x)) \wedge (n{\neq}0 \to B_{Sh}(v,y))$
$\leftrightarrow \forall u, v \exists x, y\, \big(A_{Sh}(u,x) \wedge B_{Sh}(v,y)\big).$

Proofs based on full classical logic

Negative translation N combined with D (i.e. $D \circ N$) gives:

Theorem

From a proof of

$$\mathbf{PA}^{\omega}+\mathbf{QF\text{-}AC} \;\vdash \forall \mathbf{x}^{\rho} \left(\forall \mathbf{u}\, \mathbf{B}_{\mathbf{qf}}(\mathbf{x},\mathbf{u}) \to \exists \mathbf{y}^{\tau}\, \mathbf{A}_{\mathbf{qf}}(\mathbf{x},\mathbf{y})\right)$$

one can extract by Sh a primitive recursive functionals Φ s.t.

$$\mathcal{S}^{\omega} \models \forall \mathbf{x}^{\rho} \left(\forall \mathbf{u}\, \mathbf{B}_{\mathbf{qf}}(\mathbf{x},\mathbf{u}) \to \mathbf{A}_{\mathbf{qf}}(\mathbf{x},\boldsymbol{\Phi}(\mathbf{x}))\right)$$

(ρ, τ arbitrary, A_{qf}, B_{qf} quantifier-free, QF-AC restriction of AC to quantifier-free formulas).

Comments

- The program extraction theorem scales down to weak systems such as RCA_0 (where then Φ is ordinarily prim. rec., Parsons 1971) or of bounded arithmetic (where then Φ is basic feasible, Cook/Urquhart 1993).

Comments

- The program extraction theorem scales down to weak systems such as RCA_0 (where then Φ is ordinarily prim. rec., Parsons 1971) or of bounded arithmetic (where then Φ is basic feasible, Cook/Urquhart 1993).
- It also scales up all the way to full countable and even dependent choice (including full 2nd order arithmetic), where then Φ is bar recursive: Spector 1962 (Consistency proof for analysis!).

Comments

- The program extraction theorem scales down to weak systems such as RCA_0 (where then Φ is ordinarily prim. rec., Parsons 1971) or of bounded arithmetic (where then Φ is basic feasible, Cook/Urquhart 1993).
- It also scales up all the way to full countable and even dependent choice (including full 2nd order arithmetic), where then Φ is bar recursive: Spector 1962 (Consistency proof for analysis!).
- Since 2000: more than 70 papers with new results in core mathematics using functional interpretation!

Comments

- The program extraction theorem scales down to weak systems such as RCA_0 (where then Φ is ordinarily prim. rec., Parsons 1971) or of bounded arithmetic (where then Φ is basic feasible, Cook/Urquhart 1993).
- It also scales up all the way to full countable and even dependent choice (including full 2nd order arithmetic), where then Φ is bar recursive: Spector 1962 (Consistency proof for analysis!).
- Since 2000: more than 70 papers with new results in core mathematics using functional interpretation!
- Partial alternative (used in automated program extraction): Friedman-Dragalin A-interpretation after negative translation as intermediate step (variants: Schwichtenberg, Coquand-Hofmann).

Connection to no-counterexample interpretation

Let A be a prenex (arithmetical) formula and $A^S, A^{ND}, A^{n.c.i}$ its Skolem, $D \circ N$ and $n.c.i.$ interpretations resp., then

$$\mathbf{HA^\omega \vdash A^S \rightarrow A^{ND} \rightarrow A^{n.c.i}},$$

but the converse implications in general fail to hold even in PA$^\omega$+QF-AC!

Connection to no-counterexample interpretation

Let A be a prenex (arithmetical) formula and $A^S, A^{ND}, A^{n.c.i}$ its Skolem, $D \circ N$ and $n.c.i.$ interpretations resp., then

$$\mathbf{HA^{\omega} \vdash A^S \rightarrow A^{ND} \rightarrow A^{n.c.i}},$$

but the converse implications in general fail to hold even in PA$^{\omega}$+QF-AC!

- A^S **too strong** (for a computable solution): Specker!

Connection to no-counterexample interpretation

Let A be a prenex (arithmetical) formula and $A^S, A^{ND}, A^{n.c.i}$ its Skolem, $D \circ N$ and $n.c.i.$ interpretations resp., then

$$\mathbf{HA^{\omega} \vdash A^{S} \to A^{ND} \to A^{n.c.i}},$$

but the converse implications in general fail to hold even in $\mathrm{PA}^{\omega}+\mathrm{QF\text{-}AC}$!

- A^S **too strong** (for a computable solution): Specker!
- $A^{n.c.i.}$ **too weak** (see IPP above; modus ponens problem).

Connection to no-counterexample interpretation

Let A be a prenex (arithmetical) formula and $A^S, A^{ND}, A^{n.c.i}$ its Skolem, $D \circ N$ and $n.c.i.$ interpretations resp., then

$$\mathbf{HA^\omega \vdash A^S \rightarrow A^{ND} \rightarrow A^{n.c.i}},$$

but the converse implications in general fail to hold even in PA^ω+QF-AC!

- A^S **too strong** (for a computable solution): Specker!
- $A^{n.c.i.}$ **too weak** (see IPP above; modus ponens problem).
- A^{ND} **just right**:

$$\mathbf{PA^\omega + QF\text{-}AC \vdash A \leftrightarrow A^{ND}}.$$

Majorizability

The functionals occurring in functional interpretation have a striking mathematical structure property:

Definition (W.A. Howard 1973)

$$\begin{cases} \mathbf{x}^* \gtrsim_{\mathbb{N}} \mathbf{x} :\equiv \mathbf{x}^* \geq \mathbf{x}, \\ \mathbf{x}^* \gtrsim_{\rho\to\tau} \mathbf{x} :\equiv \forall \mathbf{y}^*, \mathbf{y}(\mathbf{y}^* \gtrsim_{\rho} \mathbf{y} \to \mathbf{x}^*(\mathbf{y}^*) \gtrsim_{\tau} \mathbf{x}(\mathbf{y})). \end{cases}$$

Read: 'x^* **majorizes** x' for $x^* \gtrsim x$.

Majorizability

The functionals occurring in functional interpretation have a striking mathematical structure property:

Definition (W.A. Howard 1973)

$$\begin{cases} \mathbf{x^* \gtrsim_{\mathbb{N}} x :\equiv x^* \geq x,} \\ \mathbf{x^* \gtrsim_{\rho\to\tau} x :\equiv \forall y^*, y(y^* \gtrsim_\rho y \to x^*(y^*) \gtrsim_\tau x(y)).} \end{cases}$$

Read: 'x^* **majorizes** x' for $x^* \gtrsim x$.

Proposition (W.A. Howard 1973)

To each closed term t^ρ of PA^ω one can define a closed term t^* s.t.

$$\mathbf{HA^\omega \vdash t^* \gtrsim_\rho t.}$$

Majorizability

The functionals occurring in functional interpretation have a striking mathematical structure property:

Definition (W.A. Howard 1973)

$$\begin{cases} \mathbf{x^* \gtrsim_{\mathbb{N}} x :\equiv x^* \geq x,} \\ \mathbf{x^* \gtrsim_{\rho\to\tau} x :\equiv \forall y^*, y(y^* \gtrsim_{\rho} y \to x^*(y^*) \gtrsim_{\tau} x(y)).} \end{cases}$$

Read: 'x^* **majorizes** x' for $x^* \gtrsim x$.

Proposition (W.A. Howard 1973)

To each closed term t^ρ of PA^ω one can define a closed term t^* s.t.

$$\mathbf{HA^{\omega} \vdash t^* \gtrsim_{\rho} t.}$$

Monotone functional interpretation MD (K.96) directly extracts t^*.

Uniform bound extraction by NMD

Let Δ be a set of $\mathcal{S}^\omega$-valid sentences of the form

$$\forall \mathbf{a}^\gamma \exists \mathbf{b} \leq_\delta \mathbf{ta} \forall \mathbf{c}^\eta \, \mathbf{F_{qf}(a, b, c)}$$

(γ, δ, η arbitrary, t closed term, F_{qf} quantifier-free).

Uniform bound extraction by NMD

Let Δ be a set of $\mathcal{S}^\omega$-valid sentences of the form

$$\forall \mathbf{a}^\gamma \exists \mathbf{b} \leq_\delta \mathbf{t}\mathbf{a} \forall \mathbf{c}^\eta \, \mathbf{F}_{\mathbf{qf}}(\mathbf{a}, \mathbf{b}, \mathbf{c})$$

$(\gamma, \delta, \eta$ arbitrary, t closed term, F_{qf} quantifier-free).

Example of axiom Δ: WKL!

Uniform bound extraction by NMD

Let Δ be a set of $\mathcal{S}^\omega$-valid sentences of the form

$$\forall \mathbf{a}^\gamma \exists \mathbf{b} \leq_\delta \mathbf{ta} \forall \mathbf{c}^\eta \, \mathbf{F_{qf}(a, b, c)}$$

(γ, δ, η arbitrary, t closed term, F_{qf} quantifier-free).

Example of axiom Δ: WKL!

Theorem (K., JSL 1992)

From a proof of

$$\mathbf{PA^\omega + QF\text{-}AC + \Delta} \; \vdash \forall \mathbf{x}^{\mathbb{N}\to\mathbb{N}} \forall \mathbf{y} \leq_\rho \mathbf{sx} \, \exists \mathbf{z}^{\mathbb{N}} \, \mathbf{A_{qf}(x, y, z)}$$

one can extract by $MD \circ N$ a primitive recursive functionals Φ s.t.

$$\mathcal{S}^\omega \models \forall \mathbf{x}^{\mathbb{N}\to\mathbb{N}} \forall \mathbf{y} \leq_\rho \mathbf{sx} \, \exists \mathbf{z} \leq \mathbf{\Phi(x)} \, \mathbf{A_{qf}(x, y, z)}.$$

(ρ arbitrary, A_{qf} quantifier-free, s closed term).

General logical metatheorems I

- Context: **continuous functions** between constructively represented **Polish spaces**.

General logical metatheorems I

- Context: **continuous functions** between constructively represented **Polish spaces**.
- Uniformity w.r.t. parameters from **compact** Polish spaces.

General logical metatheorems I

- Context: **continuous functions** between constructively represented **Polish spaces**.
- Uniformity w.r.t. parameters from **compact** Polish spaces.
- Extraction of **bounds** from **noneffective** existence proofs.

In the uniform bound extraction theorem, the bound $\Phi(x)$ only depends on $x \in \mathbb{N}^{\mathbb{N}}$ but not on $y \leq_\rho sx$. For $\rho = \mathbb{N} \to \mathbb{N}$ and $s(x) :\equiv 1$ this, in particular, gives **independence from parameters in Cantor space $2^{\mathbb{N}}$**.

In the uniform bound extraction theorem, the bound $\Phi(x)$ only depends on $x \in \mathbb{N}^{\mathbb{N}}$ but not on $y \leq_\rho sx$. For $\rho = \mathbb{N} \to \mathbb{N}$ and $s(x) :\equiv 1$ this, in particular, gives **independence from parameters in Cantor space $2^{\mathbb{N}}$**.

General **Polish spaces** X and **compact metric spaces** K can be **represented** as continuous images of Baire space $\mathcal{B} := \mathbb{N}^{\mathbb{N}}$ resp. Cantor space $\mathcal{C} := 2^{\mathbb{N}}$.

In the uniform bound extraction theorem, the bound $\Phi(x)$ only depends on $x \in \mathbb{N}^{\mathbb{N}}$ but not on $y \leq_\rho sx$. For $\rho = \mathbb{N} \to \mathbb{N}$ and $s(x) :\equiv 1$ this, in particular, gives **independence from parameters in Cantor space $2^{\mathbb{N}}$**.

General **Polish spaces X** and **compact metric spaces K** can be **represented** as continuous images of Baire space $\mathcal{B} := \mathbb{N}^{\mathbb{N}}$ resp. Cantor space $\mathcal{C} := 2^{\mathbb{N}}$.

Polish spaces X are represented as the quotient of the space of fast (e.g. 2^{-n}) convergent Cauchy sequences f of elements of a countable dense subset (f can be viewed as elements in $\mathbb{N}^{\mathbb{N}}$) w.r.t. equivalence relation

$$\mathbf{f =_X g :\equiv \lim_{n\to\infty} d_X(f(n), g(n)) =_{\mathbb{R}} 0.}$$

K., 1993-96: P Polish space, K a compact P-space, $A_\exists$ existential.
BA:= **basic arithmetic**, HBC Heine/Borel compactness WKL (SEQ$^-$ restricted sequential compactness, ACA).

K., 1993-96: P Polish space, K a compact P-space, $A_\exists$ existential.
BA:= **basic arithmetic**, HBC Heine/Borel compactness WKL (SEQ^- restricted sequential compactness, ACA).
From a proof

$$\mathbf{BA + HBC(+SEQ^-)} \vdash \forall \mathbf{x} \in \mathbf{P}\, \forall \mathbf{y} \in \mathbf{K}\, \exists \mathbf{m} \in \mathbb{N}\, \mathbf{A}_\exists(\mathbf{x}, \mathbf{y}, \mathbf{m})$$

K., 1993-96: P Polish space, K a compact P-space, $A_\exists$ existential.
BA:= **basic arithmetic**, HBC Heine/Borel compactness WKL (SEQ^- restricted sequential compactness, ACA).
From a proof

$$\mathbf{BA} + \mathbf{HBC}(+\mathbf{SEQ}^-) \vdash \forall \mathbf{x} \in \mathbf{P}\, \forall \mathbf{y} \in \mathbf{K}\, \exists \mathbf{m} \in \mathbb{N}\, \mathbf{A}_\exists(\mathbf{x}, \mathbf{y}, \mathbf{m})$$

one can extract a closed term Φ of BA (+iteration)

$$\mathbf{BA}\ (+\ \mathbf{IA}) \vdash \forall \mathbf{x} \in \mathbf{P}\, \forall \mathbf{y} \in \mathbf{K}\, \exists \mathbf{m} \leq \mathbf{\Phi}(\mathbf{f_x})\, \mathbf{A}_\exists(\mathbf{x}, \mathbf{y}, \mathbf{m}).$$

K., 1993-96: P Polish space, K a compact P-space, $A_\exists$ existential.
BA:= **basic arithmetic**, HBC Heine/Borel compactness WKL (SEQ^- restricted sequential compactness, ACA).
From a proof

$$\mathbf{BA + HBC(+SEQ^-) \vdash \forall x \in P\, \forall y \in K\, \exists m \in \mathbb{N}\, A_\exists(x, y, m)}$$

one can extract a closed term Φ of BA (+iteration)

$$\mathbf{BA\ (+\ IA\ ) \vdash \forall x \in P\, \forall y \in K\, \exists m \leq \Phi(f_x)\, A_\exists(x, y, m).}$$

Important:

$\Phi(f_x)$ does **not depend** on $y \in K$ but on a **representation** f_x of x!

Limits of Metatheorem for concrete spaces

Compactness means constructively: **completeness** and **total boundedness**.

Limits of Metatheorem for concrete spaces

Compactness means constructively: **completeness** and **total boundedness**.

Necessity of completeness: The set $[0,2]_{\mathbb{Q}}$ is totally bounded and constructively representable and

$$\mathbf{BA} \vdash \forall \mathbf{q} \in [\mathbf{0},\mathbf{2}]_{\mathbb{Q}}\ \exists \mathbf{n} \in \mathbb{N}(|\mathbf{q} - \sqrt{\mathbf{2}}| >_{\mathbb{R}} \mathbf{2}^{-\mathbf{n}}).$$

However: **no uniform bound on $\exists n \in \mathbb{N}$!**

Necessity of total boundedness: Let B be the unit ball $C[0,1]$. B is bounded and constructively representable.

By Weierstraß' theorem

$$\mathbf{BA} \vdash \forall \mathbf{f} \in \mathbf{B} \exists \mathbf{n} \in \mathbb{N} \big(\mathbf{n} \text{ code of } \mathbf{p} \in \mathbb{Q}[\mathbf{X}] \text{ s.t. } \|\mathbf{p} - \mathbf{f}\|_\infty < \frac{1}{2} \big)$$

but **no uniform bound** on $\exists n$: take $f_n := \sin(nx)$.

Necessity of $A_\exists$ '$\exists$-formula':

Let (f_n) be the usual sequence of spike-functions in $C[0,1]$, s.t. (f_n) converges pointwise but not uniformly towards 0. Then

$$\mathbf{BA} \vdash \forall \mathbf{x} \in [\mathbf{0},\mathbf{1}] \forall \mathbf{k} \in \mathbb{N} \exists \mathbf{n} \in \mathbb{N} \forall \mathbf{m} \in \mathbb{N} (|\mathbf{f}_{\mathbf{n}+\mathbf{m}}(\mathbf{x})| \leq \mathbf{2}^{-\mathbf{k}}),$$

but **no uniform bound** on '$\exists n$' (proof based on Σ^0_1-LEM).

Necessity of $A_\exists$ '$\exists$-formula':

Let (f_n) be the usual sequence of spike-functions in $C[0,1]$, s.t. (f_n) converges pointwise but not uniformly towards 0. Then

$$\mathbf{BA} \vdash \forall \mathbf{x} \in [\mathbf{0},\mathbf{1}] \forall \mathbf{k} \in \mathbb{N} \exists \mathbf{n} \in \mathbb{N} \forall \mathbf{m} \in \mathbb{N} (|\mathbf{f}_{\mathbf{n}+\mathbf{m}}(\mathbf{x})| \leq \mathbf{2}^{-\mathbf{k}}),$$

but **no uniform bound** on '$\exists n$' (proof based on Σ^0_1-LEM).

Uniform bound only if $(f_n(x))$ **monotone** (Dini): **'$\forall m \in \mathbb{N}$' superfluous!**

Necessity of $\Phi(f_x)$ depending on a representative of x :

Consider

$$\mathbf{BA} \vdash \forall \mathbf{x} \in \mathbb{R} \exists \mathbf{n} \in \mathbb{N} (\mathbf{n} >_{\mathbb{R}} \mathbf{x}).$$

Suppose there would exist an $=_{\mathbb{R}}$-extensional computable $\Phi : \mathbb{N}^{\mathbb{N}} \to \mathbb{N}$ producing such a n. Then Φ would represent a **continuous** and hence **constant** function $\mathbb{R} \to \mathbb{N}$ which gives a contradiction.

Unique existence

P, K Polish, K compact, $f : P \times K \to \mathbb{R}$ (BA-definable).

Unique existence

P, K Polish, K compact, $f : P \times K \to \mathbb{R}$ (BA-definable).

MFI transforms **uniqueness statements**

$$\forall x \in P, y_1, y_2 \in K \Big(\bigwedge_{i=1}^{2} f(x, y_i) =_{\mathbb{R}} 0 \to d_K(y_1, y_2) =_{\mathbb{R}} 0 \Big)$$

into **moduli of uniqueness** $\Phi : \mathbb{Q}_+^* \to \mathbb{Q}_+^*$

$$\forall x \in P, y_1, y_2 \in K, \varepsilon > 0 \Big(\bigwedge_{i=1}^{2} |f(x, y_i)| < \Phi(x, \varepsilon) \to d_K(y_1, y_2) < \varepsilon \Big).$$

Unique existence

P, K Polish, K compact, $f : P \times K \to \mathbb{R}$ (BA-definable).

MFI transforms **uniqueness statements**

$$\forall \mathbf{x} \in \mathbf{P}, \mathbf{y_1}, \mathbf{y_2} \in \mathbf{K} \Big(\bigwedge_{i=1}^{2} \mathbf{f}(\mathbf{x}, \mathbf{y_i}) =_{\mathbb{R}} \mathbf{0} \to \mathbf{d_K}(\mathbf{y_1}, \mathbf{y_2}) =_{\mathbb{R}} \mathbf{0} \Big)$$

into **moduli of uniqueness** $\Phi : \mathbb{Q}^*_+ \to \mathbb{Q}^*_+$

$$\forall \mathbf{x} \in \mathbf{P}, \mathbf{y_1}, \mathbf{y_2} \in \mathbf{K}, \varepsilon > \mathbf{0} \Big(\bigwedge_{i=1}^{2} |\mathbf{f}(\mathbf{x}, \mathbf{y_i})| < \boldsymbol{\Phi}(\mathbf{x}, \varepsilon) \to \mathbf{d_K}(\mathbf{y_1}, \mathbf{y_2}) < \varepsilon \Big).$$

Let $\widehat{y} \in K$ be the unique root of $f(x, \cdot)$, y_ε an ε-root $|f(x, y_\varepsilon)| < \varepsilon$.

Unique existence

P, K Polish, K compact, $f : P \times K \to \mathbb{R}$ (BA-definable).

MFI transforms **uniqueness statements**

$$\forall \mathbf{x} \in \mathbf{P}, \mathbf{y_1}, \mathbf{y_2} \in \mathbf{K} \Big(\bigwedge_{i=1}^{2} \mathbf{f}(\mathbf{x}, \mathbf{y_i}) =_{\mathbb{R}} \mathbf{0} \to \mathbf{d_K}(\mathbf{y_1}, \mathbf{y_2}) =_{\mathbb{R}} \mathbf{0} \Big)$$

into **moduli of uniqueness** $\Phi : \mathbb{Q}^*_+ \to \mathbb{Q}^*_+$

$$\forall \mathbf{x} \in \mathbf{P}, \mathbf{y_1}, \mathbf{y_2} \in \mathbf{K}, \varepsilon > \mathbf{0} \Big(\bigwedge_{i=1}^{2} |\mathbf{f}(\mathbf{x}, \mathbf{y_i})| < \boldsymbol{\Phi}(\mathbf{x}, \varepsilon) \to \mathbf{d_K}(\mathbf{y_1}, \mathbf{y_2}) < \varepsilon \Big).$$

Let $\widehat{y} \in K$ be the unique root of $f(x, \cdot)$, y_ε an ε-root $|f(x, y_\varepsilon)| < \varepsilon$.
Then

$$\mathbf{d_K}(\widehat{\mathbf{y}}, \mathbf{y}_{\boldsymbol{\Phi}(\mathbf{x},\varepsilon)}) < \varepsilon).$$

Literature

1) Avigad, J., Feferman, S., Gödel's functional ('Dialectica') interpretation. In: Buss, S. (ed.), Handbook of Proof Theory, North-Holland, pp. 337-405 (1998).

2) Ferreira, F., Oliva, P., Bounded functional interpretation. Ann. Pure Appl. Logic **135**, pp. 73-112 (2005).

3) Gödel, K., Über eine bisher noch nicht benützte Erweiterung des finiten Standpunktes. Dialectica **12**, pp. 280-287 (1958).

4) Howard, W.A., Hereditarily majorizable functionals of finite type. In: Troelstra (ed.), Metamathematical investigation of intuitionistic arithmetic and analysis, pp. 454-461. Springer LNM 344 (1973).

5) Kohlenbach, U., Effective bounds from ineffective proofs in analysis: an application of functional interpretation and majorization. J. Symbolic Logic **57**, pp. 1239–1273 (1992).

6) Kohlenbach, U., Effective moduli from ineffective uniqueness proofs. An unwinding of de La Vallée Poussin's proof for Chebycheff approximation. Ann. Pure Appl. Logic **64**, pp. 27-94 (1993).

7) Kohlenbach, U., Analysing proofs in analysis. In: W. Hodges, M. Hyland, C. Steinhorn, J. Truss, editors, *Logic: from Foundations to Applications. European Logic Colloquium* (Keele, 1993), pp. 225–260, Oxford University Press (1996).

8) Kohlenbach, U., On the no-counterexample interpretation. J. Symbolic Logic **64**, pp. 1491-1511 (1999).

9) Kohlenbach, U., Applied Proof Theory: Proof Interpretations and their Use in Mathematics. Springer Monographs in Mathematics. xx+536pp., Springer Heidelberg-Berlin, 2008.

10) Kohlenbach, U., Gödel's functional interpretation and its use in current mathematics. In: Kurt Gödel and the Foundations of Mathematics. Horizons of Truth. Baaz, M. et al. (eds.), Cambridge University Press, pp. 361-398, New York 2011. Reprinted in: dialectica Vol. 62, no. 2, pp. 223-267 (2008).

11) Kohlenbach, U., Oliva, P., Proof mining: a systematic way of analysing proofs in mathematics. Proc. Steklov Inst. Math. **242**, pp. 136-164 (2003).

12) Kreisel, G., On the interpretation of non-finitist proofs, part I. J. Symbolic Logic **16**, pp.241-267 (1951).

13) Kreisel, G., On the interpretation of non-finitist proofs, part II: Interpretation of number theory, applications. J. Symbolic Logic **17**, pp. 43-58 (1952).

14) Luckhardt, H., Extensional Gödel Functional Interpretation. Springer LNM **306**, 1973.

15) Mints, G.E., Theory of Proofs (Arithmetic and Analysis). J. Soviet Math. **7**, pp. 501-531 (1977) (Translation from 1975 Russian).

16) Spector, C., Provably recursive functionals of analysis: a consistency proof of analysis by an extension of principles formulated in current intuitionistic mathematics. In: Recursive function theory, Proceedings of Symposia in Pure Mathematics, vol. 5 (J.C.E. Dekker (ed.)), AMS, Providence, R.I., pp. 1–27 (1962).

17) Troelstra, A.S. (ed.) Metamathematical investigation of intuitionistic arithmetic and analysis. Springer LNM **344** (1973).

Lecture III

Unique existence

P, K Polish, K compact, $f : P \times K \to \mathbb{R}$ (BA-definable).

Unique existence

P, K Polish, K compact, $f : P \times K \to \mathbb{R}$ (BA-definable).

MFI transforms **uniqueness statements**

$$\forall x \in P, y_1, y_2 \in K \left(\bigwedge_{i=1}^{2} f(x, y_i) =_{\mathbb{R}} 0 \to d_K(y_1, y_2) =_{\mathbb{R}} 0 \right)$$

into **moduli of uniqueness** $\Phi : \mathbb{Q}_+^* \to \mathbb{Q}_+^*$

$$\forall x \in P, y_1, y_2 \in K, \varepsilon > 0 \left(\bigwedge_{i=1}^{2} |f(x, y_i)| < \Phi(x, \varepsilon) \to d_K(y_1, y_2) < \varepsilon \right).$$

Unique existence

P, K Polish, K compact, $f : P \times K \to \mathbb{R}$ (BA-definable).

MFI transforms **uniqueness statements**

$$\forall \mathbf{x} \in \mathbf{P}, \mathbf{y_1}, \mathbf{y_2} \in \mathbf{K} \Big(\bigwedge_{\mathbf{i}=1}^{2} \mathbf{f}(\mathbf{x}, \mathbf{y_i}) =_{\mathbb{R}} \mathbf{0} \to \mathbf{d_K}(\mathbf{y_1}, \mathbf{y_2}) =_{\mathbb{R}} \mathbf{0} \Big)$$

into **moduli of uniqueness** $\Phi : \mathbb{Q}^*_+ \to \mathbb{Q}^*_+$

$$\forall \mathbf{x} \in \mathbf{P}, \mathbf{y_1}, \mathbf{y_2} \in \mathbf{K}, \varepsilon > \mathbf{0} \Big(\bigwedge_{\mathbf{i}=1}^{2} |\mathbf{f}(\mathbf{x}, \mathbf{y_i})| < \mathbf{\Phi}(\mathbf{x}, \varepsilon) \to \mathbf{d_K}(\mathbf{y_1}, \mathbf{y_2}) < \varepsilon \Big).$$

Let $\widehat{y} \in K$ be the unique root of $f(x, \cdot)$, y_ε an ε-root $|f(x, y_\varepsilon)| < \varepsilon$.

Unique existence

P, K Polish, K compact, $f : P \times K \to \mathbb{R}$ (BA-definable).

MFI transforms **uniqueness statements**

$$\forall \mathbf{x} \in \mathbf{P}, \mathbf{y_1}, \mathbf{y_2} \in \mathbf{K} \left(\bigwedge_{\mathbf{i}=1}^{2} \mathbf{f}(\mathbf{x}, \mathbf{y_i}) =_{\mathbb{R}} \mathbf{0} \to \mathbf{d_K}(\mathbf{y_1}, \mathbf{y_2}) =_{\mathbb{R}} \mathbf{0} \right)$$

into **moduli of uniqueness** $\Phi : \mathbb{Q}^*_+ \to \mathbb{Q}^*_+$

$$\forall \mathbf{x} \in \mathbf{P}, \mathbf{y_1}, \mathbf{y_2} \in \mathbf{K}, \varepsilon > \mathbf{0} \left(\bigwedge_{\mathbf{i}=1}^{2} |\mathbf{f}(\mathbf{x}, \mathbf{y_i})| < \boldsymbol{\Phi}(\mathbf{x}, \varepsilon) \to \mathbf{d_K}(\mathbf{y_1}, \mathbf{y_2}) < \varepsilon \right).$$

Let $\widehat{y} \in K$ be the unique root of $f(x, \cdot)$, y_ε an ε-root $|f(x, y_\varepsilon)| < \varepsilon$. Then

$$\mathbf{d_K}(\widehat{\mathbf{y}}, \mathbf{y}_{\boldsymbol{\Phi}(\mathbf{x},\varepsilon)}) < \varepsilon).$$

Case study: strong unicity in L_1-approximation

P_n space of polynomials of degree $\leq n$, $f \in C[0,1]$,

$$\|f\|_1 := \int_0^1 |f(x)|dx, \quad \text{dist}_1(f, P_n) := \inf_{p \in P_n} \|f - p\|_1.$$

Case study: strong unicity in L_1-approximation

P_n space of polynomials of degree $\leq n$, $f \in C[0,1]$,

$$\|\mathbf{f}\|_1 := \int_0^1 |\mathbf{f}(\mathbf{x})|\mathbf{dx}, \quad \mathbf{dist_1}(\mathbf{f}, \mathbf{P_n}) := \inf_{\mathbf{p} \in \mathbf{P_n}} \|\mathbf{f} - \mathbf{p}\|_1.$$

Best **approximation in the mean** of $f \in C[0,1]$ (Jackson 1926):

$$\forall \mathbf{f} \in \mathbf{C}[0,1] \exists! \mathbf{p_b} \in \mathbf{P_n} \big(\|\mathbf{f} - \mathbf{p_b}\|_1 = \mathbf{dist_1}(\mathbf{f}, \mathbf{P_n}) \big)$$

(existence **and** uniqueness use: WKL!)

Logical Pre-Processing I

In order to apply our metatheorem on uniqueness proofs we restrict $\mathbf{P_n}$ to the **compact** subset $\mathbf{K_{f,n} := \{p \in P_n \ : \ \|p\|_1 \leq \frac{5}{2}\|f\|_1\}}$.

Logical Pre-Processing I

In order to apply our metatheorem on uniqueness proofs we restrict $\mathbf{P_n}$ to the **compact** subset $\mathbf{K_{f,n} := \{p \in P_n \ : \ \|p\|_1 \leq \frac{5}{2}\|f\|_1\}}$.

Let $\mathbf{\Phi(f, n, \varepsilon)}$ be a modulus of uniqueness on $\mathbf{K_{f,n}}$. Then

$$\tilde{\Phi}(f, n, \varepsilon) := \min\left(\frac{\varepsilon}{8}, \Phi(f, n, \varepsilon)\right)$$

is a modulus of uniqueness on all of $\mathbf{P_n}$:

Logical Pre-Processing I

In order to apply our metatheorem on uniqueness proofs we restrict $\mathbf{P_n}$ to the **compact** subset $\mathbf{K_{f,n} := \{p \in P_n \ : \ \|p\|_1 \leq \frac{5}{2}\|f\|_1\}}$.

Let $\mathbf{\Phi(f, n, \varepsilon)}$ be a modulus of uniqueness on $\mathbf{K_{f,n}}$. Then

$$\tilde{\Phi}(f, n, \varepsilon) := \min\left(\frac{\varepsilon}{8}, \Phi(f, n, \varepsilon)\right)$$

is a modulus of uniqueness on all of $\mathbf{P_n}$:

Let $\mathbf{p_1 \in P_n / K_{f,n}}$. Then $\mathbf{\|f - p_1\|_1 > \frac{3}{2}\|f\|_1 \geq \frac{3}{2}dist_1(f, P_n)}$ since $\mathbf{0 \in P_n}$. Assume

$$\|f - p_1\|_1, \|f - p_2\|_1 < dist_1(f, P_n) + \tilde{\Phi}(f, n, \varepsilon) \leq dist_1(f, P_n) + \frac{\varepsilon}{8}.$$

Then $\mathbf{dist_1(f, P_n) < \frac{\varepsilon}{4}}$ and so

$$\|p_1 - p_2\|_1 \leq \|f - p_1\|_1 + \|f - p_2\|_1 < \frac{\varepsilon}{4} + \frac{\varepsilon}{8} + \frac{\varepsilon}{4} + \frac{\varepsilon}{8} < \varepsilon.$$

Logical Pre-Processing II

The **Cauchy-representation** of $f \in C[0,1$ is equivalent to f given as pair $(f_r, \omega^{\mathbb{N}\to\mathbb{N}})$, where f_r is the restriction of f to the rational numbers in $[0,1]$ (so f_r can be encoded as function $\mathbb{N} \to \mathbb{N}$) and ω is a **modulus of uniform continuity** of f

$$\forall \mathbf{k} \in \mathbb{N}\, \forall \mathbf{x}, \mathbf{y} \in [\mathbf{0}, \mathbf{1}]\, \big(|\mathbf{x} - \mathbf{y}| \leq \mathbf{2}^{-\omega(\mathbf{k})} \to |\mathbf{f}(\mathbf{x}) - \mathbf{f}(\mathbf{y})| \leq \mathbf{2}^{-\mathbf{k}}\big).$$

Logical Pre-Processing II

The **Cauchy-representation** of $f \in C[0,1$ is equivalent to f given as pair $(f_r, \omega^{\mathbb{N}\to\mathbb{N}})$, where f_r is the restriction of f to the rational numbers in $[0,1]$ (so f_r can be encoded as function $\mathbb{N} \to \mathbb{N}$) and ω is a **modulus of uniform continuity** of f

$$\forall \mathbf{k} \in \mathbb{N}\, \forall \mathbf{x}, \mathbf{y} \in [\mathbf{0}, \mathbf{1}] \left(|\mathbf{x} - \mathbf{y}| \le \mathbf{2}^{-\boldsymbol{\omega}(\mathbf{k})} \to |\mathbf{f}(\mathbf{x}) - \mathbf{f}(\mathbf{y})| \le \mathbf{2}^{-\mathbf{k}}\right).$$

Given $\mathbb{N} \ni M \ge \|f\|_\infty$ (easily computable in (f_r, ω)), f_r can be **majorized**. Hence Φ can be arranged to only depend on M and ω.

Logical Pre-Processing II

The **Cauchy-representation** of $f \in C[0,1$ is equivalent to f given as pair $(f_r, \omega^{\mathbb{N}\to\mathbb{N}})$, where f_r is the restriction of f to the rational numbers in $[0,1]$ (so f_r can be encoded as function $\mathbb{N} \to \mathbb{N}$) and ω is a **modulus of uniform continuity** of f

$$\forall \mathbf{k} \in \mathbb{N}\, \forall \mathbf{x}, \mathbf{y} \in [\mathbf{0}, \mathbf{1}] \left(|\mathbf{x} - \mathbf{y}| \leq \mathbf{2}^{-\omega(\mathbf{k})} \to |\mathbf{f}(\mathbf{x}) - \mathbf{f}(\mathbf{y})| \leq \mathbf{2}^{-\mathbf{k}} \right).$$

Given $\mathbb{N} \ni M \geq \|f\|_\infty$ (easily computable in (f_r, ω)), f_r can be **majorized**. Hence Φ can be arranged to only depend on M and ω.

Replacing f by $\tilde{f}(x) := f(x) - f(0)$ one can assume w.l.o.g. that $M := 2^{\omega(0)}$ does the job. So we know **a priorily** the extractability of a prim. rec. (in the sense of Hilbert-Gödel) modulus of uniqueness which **only depends** on ε, n and ω!

Theorem (K./Paulo Oliva, APAL 2003)

Let $dist_1(f, P_n) := \inf_{p \in P_n} \|f - p\|_1$ and ω a modulus of uniform continuity for f.

$$\mathbf{\Psi(\omega, n, \varepsilon) := \min\{\frac{c_n\varepsilon}{8(n+1)^2}, \frac{c_n\varepsilon}{2}\omega_n(\frac{c_n\varepsilon}{2})\},\ \text{where}}$$

$$\mathbf{c_n := \frac{\lfloor n/2 \rfloor ! \lceil n/2 \rceil !}{2^{4n+3}(n+1)^{3n+1}}\ \text{and}}$$

$$\mathbf{\omega_n(\varepsilon) := \min\{\omega(\frac{\varepsilon}{4}), \frac{\varepsilon}{40(n+1)^4 \lceil \frac{1}{\omega(1)} \rceil}\}.}$$

Theorem (K./Paulo Oliva, APAL 2003)

Let $dist_1(f, P_n) := \inf_{p \in P_n} \|f - p\|_1$ and ω a modulus of uniform continuity for f.

$$\mathbf{\Psi(\omega, n, \varepsilon) := \min\{\tfrac{c_n \varepsilon}{8(n+1)^2}, \tfrac{c_n \varepsilon}{2}\omega_n(\tfrac{c_n \varepsilon}{2})\}}, \text{ where}$$
$$\mathbf{c_n := \frac{\lfloor n/2 \rfloor ! \lceil n/2 \rceil !}{2^{4n+3}(n+1)^{3n+1}}} \text{ and}$$
$$\mathbf{\omega_n(\varepsilon) := \min\{\omega(\tfrac{\varepsilon}{4}), \frac{\varepsilon}{40(n+1)^4 \lceil \frac{1}{\omega(1)} \rceil}\}}.$$

Then $\forall n \in \mathbb{N}, p_1, p_2 \in P_n$

$$\forall \varepsilon \in \mathbb{Q}^*_+ \Big(\bigwedge_{i=1}^{2} (\|f - p_i\|_1 - dist_1(f, P_n) \le \Psi(\omega, n, \varepsilon)) \to \|p_1 - p_2\|_1 \le \varepsilon \Big).$$

Comments on the result in the L_1-case

- Φ provides the **first effective version** of results due to Bjoernestal (1975) and Kroó (1978-1981).

Comments on the result in the L_1-case

- Φ provides the **first effective version** of results due to Bjoernestal (1975) and Kroó (1978-1981).
- Kroó (1978) implies that the ε-dependency in Φ is **optimal**.

Comments on the result in the L_1-case

- Φ provides the **first effective version** of results due to Bjoernestal (1975) and Kroó (1978-1981).
- Kroó (1978) implies that the ε-dependency in Φ is **optimal**.
- Φ is also a **modulus of pointwise continuity** of the **projection operator**.

Comments on the result in the L_1-case

- Φ provides the **first effective version** of results due to Bjoernestal (1975) and Kroó (1978-1981).
- Kroó (1978) implies that the ε-dependency in Φ is **optimal**.
- Φ is also a **modulus of pointwise continuity** of the **projection operator**.
- Φ gave the **first complexity upper bound** for the sequence of best L_1-approximations (p_n) in P_n of poly-time functions $f \in C[0,1]$ (P. Oliva, MLQ 2003).

Comments on the result in the L_1-case

- Φ provides the **first effective version** of results due to Bjoernestal (1975) and Kroó (1978-1981).
- Kroó (1978) implies that the ε-dependency in Φ is **optimal**.
- Φ is also a **modulus of pointwise continuity** of the **projection operator**.
- Φ gave the **first complexity upper bound** for the sequence of best L_1-approximations (p_n) in P_n of poly-time functions $f \in C[0,1]$ (P. Oliva, MLQ 2003).

Many other applications to best Chebycheff Approximation (i.e. best approximation w.r.t. the uniform norm $\|\cdot\|_\infty$).

The nonseparable/noncompact case: a simply example

Proposition

Let $(X, \|\cdot\|)$ be a strictly convex normed space and $C \subseteq X$ a convex subset. Then any point $x \in X$ has at most one point $c \in C$ of minimal distance, i.e. $\|x - c\| = \text{dist}(x, C)$.

The nonseparable/noncompact case: a simply example

Proposition

Let $(X, \|\cdot\|)$ be a strictly convex normed space and $C \subseteq X$ a convex subset. Then any point $x \in X$ has at most one point $c \in C$ of minimal distance, i.e. $\|x - c\| =$dist(x, C).

Hence: if X is separable and complete and provably strictly convex and C compact, then one can extract a modulus of uniqueness.

The nonseparable/noncompact case: a simply example

Proposition

Let $(X, \|\cdot\|)$ be a strictly convex normed space and $C \subseteq X$ a convex subset. Then any point $x \in X$ has at most one point $c \in C$ of minimal distance, i.e. $\|x - c\| =$dist(x, C).

Hence: if X is separable and complete and provably strictly convex and C compact, then one can extract a modulus of uniqueness.

Observation: compactness only used to extract uniform bound on strict convexity (= **modulus of uniform convexity**) from proof of strict convexity.

Assume that X is uniformly convex with modulus η.

Assume that X is uniformly convex with modulus η.

Then for $d \geq \text{dist}(x, C)$ we have the following modulus of uniqueness (K.1990):

Assume that X is uniformly convex with modulus η.

Then for $d \geq$dist(x, C) we have the following modulus of uniqueness (K.1990):

$$\boldsymbol{\Phi(\varepsilon) := \min\left(1, \frac{\varepsilon}{4}, \frac{\varepsilon}{4} \cdot \frac{\eta(\varepsilon/(d+1))}{1 - \eta(\varepsilon/(d+1))}\right).}$$

Conclusion: neither compactness nor separability required!

Assume that X is uniformly convex with modulus η.

Then for $d \geq$dist(x, C) we have the following modulus of uniqueness (K.1990):

$$\mathbf{\Phi(\varepsilon) := \min\left(1, \frac{\varepsilon}{4}, \frac{\varepsilon}{4} \cdot \frac{\eta(\varepsilon/(d+1))}{1-\eta(\varepsilon/(d+1))}\right).}$$

Conclusion: neither compactness nor separability required!

In particular: **existence** of solution (for complete X and closed C) from **uniform uniqueness** which in turn stems from **uniform convexity**.

General logical metatheorems II

Many abstract types of metric structures can be added as atoms: metric, hyperbolic, CAT(0), δ-hyperbolic, normed, uniformly convex, Hilbert, abstract L^p and $C(K)$-spaces... spaces or $\mathbb{R}$-trees X : add **new base type** X, all **finite types over** $\mathbb{N}, X$ and a new **constant** d_X representing d etc.

General logical metatheorems II

Many abstract types of metric structures can be added as atoms: metric, hyperbolic, CAT(0), δ-hyperbolic, normed, uniformly convex, Hilbert, abstract L^p and $C(K)$-spaces... spaces or $\mathbb{R}$-trees X : add **new base type** X, all **finite types over** $\mathbb{N}, X$ and a new **constant** d_X representing d etc.

Condition: Defining axioms must have a monotone functional interpretation.

General logical metatheorems II

Many abstract types of metric structures can be added as atoms: metric, hyperbolic, CAT(0), δ-hyperbolic, normed, uniformly convex, Hilbert, abstract L^p and $C(K)$-spaces... spaces or $\mathbb{R}$-trees X : add **new base type** X, all **finite types over** $\mathbb{N}, X$ and a new **constant** d_X representing d etc.

Condition: Defining axioms must have a monotone functional interpretation.

Counterexamples (to extractibility of uniform bounds): for the classes of strictly convex ($\rightarrow$ uniformly convex) or separable ($\rightarrow$ totally bounded) spaces!

Formal systems for analysis with abstract spaces X

Types: (i) $\mathbb{N}, X$ are types, (ii) with ρ, τ also $\rho \to \tau$ is a type.

Functionals of type $\rho \to \tau$ map type-ρ objects to type-τ objects.

Formal systems for analysis with abstract spaces X

Types: (i) $\mathbb{N}, X$ are types, (ii) with ρ, τ also $\rho \to \tau$ is a type.

Functionals of type $\rho \to \tau$ map type-ρ objects to type-τ objects.

PA$^{\omega,X}$ is the extension of Peano Arithmetic to all types.

$\mathcal{A}^{\omega,X}$**:=PA**$^{\omega,X}$+**DC**, where

DC: axiom of dependent choice for all types

Implies **full comprehension** for numbers (higher order arithmetic).

Formal systems for analysis with abstract spaces X

Types: (i) $\mathbb{N}, X$ are types, (ii) with ρ, τ also $\rho \to \tau$ is a type.

Functionals of type $\rho \to \tau$ map type-ρ objects to type-τ objects.

PA$^{\omega,X}$ is the extension of Peano Arithmetic to all types.

$\mathcal{A}^{\omega,X}$**:=PA**$^{\omega,X}$+**DC**, where

DC: axiom of dependent choice for all types

Implies **full comprehension** for numbers (higher order arithmetic).

$\mathcal{A}^\omega[X, d, \ldots]$ results by adding constants $d_X, \ldots$ with axioms expressing that $(X, d, \ldots)$ is a nonempty metric, hyperbolic ... space.

A warning concerning equality

Extensionality rule (**only!**):

$$\frac{s =_\rho t}{r(s) =_\tau r(t)},$$

where only $x =_{\mathbb{N}} y$ primitive equality predicate but for $\rho \to \tau$

$$x^X =_X y^X :\equiv d_X(x, y) =_{\mathbb{R}} 0_{\mathbb{R}},$$
$$x =_{\rho\to\tau} y :\equiv \forall v^\rho (s(v) =_\tau t(v)).$$

A novel form of majorization

y, x functionals of types $\rho, \widehat{\rho} := \rho[\mathbb{N}/X]$ and a^X of type X:

$$\mathbf{x}^{\mathbb{N}} \gtrsim^{\mathbf{a}}_{\mathbb{N}} \mathbf{y}^{\mathbb{N}} :\equiv \mathbf{x} \geq \mathbf{y}$$
$$\mathbf{x}^{\mathbb{N}} \gtrsim^{\mathbf{a}}_{\mathbf{X}} \mathbf{y}^{\mathbf{X}} :\equiv \mathbf{x} \geq \mathbf{d}(\mathbf{y}, \mathbf{a}).$$

A novel form of majorization

y, x functionals of types $\rho, \widehat{\rho} := \rho[\mathbb{N}/X]$ and a^X of type X:

$$\mathbf{x}^{\mathbb{N}} \gtrsim^{\mathbf{a}}_{\mathbb{N}} \mathbf{y}^{\mathbb{N}} :\equiv \mathbf{x} \geq \mathbf{y}$$
$$\mathbf{x}^{\mathbb{N}} \gtrsim^{\mathbf{a}}_{\mathbf{X}} \mathbf{y}^{\mathbf{X}} :\equiv \mathbf{x} \geq \mathbf{d(y, a)}.$$

For **complex types** $\rho \to \tau$ this is extended in a **hereditary fashion**.

A novel form of majorization

y, x functionals of types $\rho, \widehat{\rho} := \rho[\mathbb{N}/X]$ and a^X of type X:

$$\mathbf{x}^{\mathbb{N}} \gtrsim^{\mathbf{a}}_{\mathbb{N}} \mathbf{y}^{\mathbb{N}} :\equiv \mathbf{x} \geq \mathbf{y}$$
$$\mathbf{x}^{\mathbb{N}} \gtrsim^{\mathbf{a}}_{\mathbf{X}} \mathbf{y}^{\mathbf{X}} :\equiv \mathbf{x} \geq \mathbf{d}(\mathbf{y}, \mathbf{a}).$$

For **complex types** $\rho \to \tau$ this is extended in a **hereditary fashion**.

Example:

$$\mathbf{f}^* \gtrsim^{\mathbf{a}}_{\mathbf{X}\to\mathbf{X}} \mathbf{f} \equiv \forall \mathbf{n} \in \mathbb{N}, \mathbf{x} \in \mathbf{X}[\mathbf{n} \geq \mathbf{d}(\mathbf{a}, \mathbf{x}) \to \mathbf{f}^*(\mathbf{n}) \geq \mathbf{d}(\mathbf{a}, \mathbf{f}(\mathbf{x}))].$$

A novel form of majorization

y, x functionals of types $\rho, \widehat{\rho} := \rho[\mathbb{N}/X]$ and a^X of type X:

$$\mathbf{x}^{\mathbb{N}} \gtrsim^{\mathbf{a}}_{\mathbb{N}} \mathbf{y}^{\mathbb{N}} :\equiv \mathbf{x} \geq \mathbf{y}$$
$$\mathbf{x}^{\mathbb{N}} \gtrsim^{\mathbf{a}}_{\mathbf{X}} \mathbf{y}^{\mathbf{X}} :\equiv \mathbf{x} \geq \mathbf{d(y, a)}.$$

For **complex types** $\rho \to \tau$ this is extended in a **hereditary fashion**.

Example:

$$\mathbf{f}^* \gtrsim^{\mathbf{a}}_{\mathbf{X} \to \mathbf{X}} \mathbf{f} \equiv \forall \mathbf{n} \in \mathbb{N}, \mathbf{x} \in \mathbf{X}[\mathbf{n} \geq \mathbf{d(a, x)} \to \mathbf{f}^*(\mathbf{n}) \geq \mathbf{d(a, f(x))}].$$

$f : X \to X$ is **nonexpansive (n.e.)** if $\mathbf{d(f(x), f(y)) \leq d(x, y)}$.

Then $\lambda n.n + b \gtrsim^{a}_{X \to X} f$, if $d(a, f(a)) \leq b$.

A novel form of majorization

y, x functionals of types $\rho, \widehat{\rho} := \rho[\mathbb{N}/X]$ and a^X of type X:

$$\mathbf{x}^{\mathbb{N}} \gtrsim^{\mathbf{a}}_{\mathbb{N}} \mathbf{y}^{\mathbb{N}} :\equiv \mathbf{x} \geq \mathbf{y}$$
$$\mathbf{x}^{\mathbb{N}} \gtrsim^{\mathbf{a}}_{\mathbf{X}} \mathbf{y}^{\mathbf{X}} :\equiv \mathbf{x} \geq \mathbf{d}(\mathbf{y}, \mathbf{a}).$$

For **complex types** $\rho \to \tau$ this is extended in a **hereditary fashion**.

Example:

$$\mathbf{f}^* \gtrsim^{\mathbf{a}}_{\mathbf{X}\to\mathbf{X}} \mathbf{f} \equiv \forall \mathbf{n} \in \mathbb{N}, \mathbf{x} \in \mathbf{X}[\mathbf{n} \geq \mathbf{d}(\mathbf{a}, \mathbf{x}) \to \mathbf{f}^*(\mathbf{n}) \geq \mathbf{d}(\mathbf{a}, \mathbf{f}(\mathbf{x}))].$$

$f : X \to X$ is **nonexpansive (n.e.)** if $\mathbf{d}(\mathbf{f}(\mathbf{x}), \mathbf{f}(\mathbf{y})) \leq \mathbf{d}(\mathbf{x}, \mathbf{y})$.

Then $\lambda n.n + b \gtrsim^a_{X\to X} f$, if $d(a, f(a)) \leq b$.

Normed linear case: $a := 0_X$.

Hyperbolic spaces

Definition (K. 2008, based on Takahashi,Kirk,Reich)

A **hyperbolic space** is a triple (X, d, W) where (X, d) is metric space and $W : X \times X \times [0, 1] \to X$ s.t.

(i) $d(z, W(x, y, \lambda)) \leq (1 - \lambda)d(z, x) + \lambda d(z, y)$,

(ii) $d(W(x, y, \lambda), W(x, y, \tilde{\lambda})) = |\lambda - \tilde{\lambda}| \cdot d(x, y)$,

(iii) $W(x, y, \lambda) = W(y, x, 1 - \lambda)$,

(iv) $d(W(x, z, \lambda), W(y, w, \lambda)) \leq (1 - \lambda)d(x, y) + \lambda d(z, w)$.

- **CAT(0)-spaces (Gromov)** are hyperbolic spaces (X, d, W) which satisfy the **CN**-inequality of Bruhat-Tits

$$\begin{cases} d(y_0, y_1) = \frac{1}{2}d(y_1, y_2) = d(y_0, y_2) \to \\ d(x, y_0)^2 \leq \frac{1}{2}d(x, y_1)^2 + \frac{1}{2}d(x, y_2)^2 - \frac{1}{4}d(y_1, y_2)^2. \end{cases}$$

- **CAT(0)-spaces (Gromov)** are hyperbolic spaces (X, d, W) which satisfy the **CN**-inequality of Bruhat-Tits

$$\begin{cases} d(y_0, y_1) = \frac{1}{2}d(y_1, y_2) = d(y_0, y_2) \to \\ d(x, y_0)^2 \le \frac{1}{2}d(x, y_1)^2 + \frac{1}{2}d(x, y_2)^2 - \frac{1}{4}d(y_1, y_2)^2. \end{cases}$$

- **convex subsets of normed spaces** = hyperbolic spaces (X, d, W) with two additional axioms (Machado (1973).

- **CAT(0)-spaces (Gromov)** are hyperbolic spaces (X,d,W) which satisfy the **CN**-inequality of Bruhat-Tits

$$\left\{ \begin{array}{l} d(y_0,y_1)=\frac{1}{2}d(y_1,y_2)=d(y_0,y_2)\rightarrow \\ d(x,y_0)^2\le \frac{1}{2}d(x,y_1)^2+\frac{1}{2}d(x,y_2)^2-\frac{1}{4}d(y_1,y_2)^2. \end{array}\right.$$

- **convex subsets of normed spaces** $=$ hyperbolic spaces (X,d,W) with two additional axioms (Machado (1973).

Notation: $(1-\lambda)x\oplus\lambda y := W(x,y,\lambda)$.

- **CAT(0)-spaces (Gromov)** are hyperbolic spaces (X, d, W) which satisfy the **CN**-inequality of Bruhat-Tits

$$\begin{cases} d(y_0, y_1) = \frac{1}{2}d(y_1, y_2) = d(y_0, y_2) \to \\ d(x, y_0)^2 \leq \frac{1}{2}d(x, y_1)^2 + \frac{1}{2}d(x, y_2)^2 - \frac{1}{4}d(y_1, y_2)^2. \end{cases}$$

- **convex subsets of normed spaces** $=$ hyperbolic spaces (X, d, W) with two additional axioms (Machado (1973).

Notation: $(1-\lambda)x \oplus \lambda y := W(x, y, \lambda)$.

Small types (over $\mathbb{N}, X$): $\mathbb{N}$, $\mathbb{N} \to \mathbb{N}$, X, $\mathbb{N} \to X$, $X \to X$.

Theorem (Gerhardy/K.,Trans.Amer.Math.Soc. 2008)

Let P, K be Polish resp. compact metric spaces, $A_\exists$ $\exists$-formula, $\underline{\tau}$ small. If $\mathcal{A}^\omega[X, d, W]$ **proves**

$$\forall \mathbf{x} \in \mathbf{P} \forall \mathbf{y} \in \mathbf{K} \forall \underline{\mathbf{z}}^{\underline{\tau}} \exists \mathbf{v}^{\mathbb{N}} \mathbf{A}_\exists(\mathbf{x}, \mathbf{y}, \underline{\mathbf{z}}, \mathbf{v}),$$

then one can extract a **computable** $\Phi : \mathbb{N}^{\mathbb{N}} \times \underline{\mathbb{N}}^{(\mathbb{N})} \to \mathbb{N}$ s.t. the following holds in every nonempty hyperbolic space: for all representatives $r_x \in \mathbb{N}^{\mathbb{N}}$ of $x \in P$ and all $\underline{z}^{\underline{\tau}}$ and $\underline{z}^* \in \mathbb{N}^{(\mathbb{N})}$ s.t. $\exists a \in X(\underline{z}^* \gtrsim^a_{\underline{\tau}} \underline{z})$:

$$\forall \mathbf{y} \in \mathbf{K} \exists \mathbf{v} \leq \mathbf{\Phi}(\mathbf{r_x}, \underline{\mathbf{z}}^*)\, \mathbf{A}_\exists(\mathbf{x}, \mathbf{y}, \underline{\mathbf{z}}, \mathbf{v}).$$

Theorem (Gerhardy/K.,Trans.Amer.Math.Soc. 2008)

Let P, K be Polish resp. compact metric spaces, $A_\exists$ $\exists$-formula, $\underline{\tau}$ small. If $\mathcal{A}^\omega[X, d, W]$ **proves**

$$\forall \mathbf{x} \in \mathbf{P} \forall \mathbf{y} \in \mathbf{K} \forall \underline{\mathbf{z}}^{\underline{\tau}} \exists \mathbf{v}^{\mathbb{N}} \mathbf{A}_\exists(\mathbf{x}, \mathbf{y}, \underline{\mathbf{z}}, \mathbf{v}),$$

then one can extract a **computable** $\Phi : \mathbb{N}^{\mathbb{N}} \times \underline{\mathbb{N}}^{(\mathbb{N})} \to \mathbb{N}$ s.t. the following holds in every nonempty hyperbolic space: for all representatives $r_x \in \mathbb{N}^{\mathbb{N}}$ of $x \in P$ and all $\underline{z}^{\underline{\tau}}$ and $\underline{z}^* \in \mathbb{N}^{(\mathbb{N})}$ s.t. $\exists a \in X(\underline{z}^* \gtrsim^a_{\underline{\tau}} \underline{z})$:

$$\forall \mathbf{y} \in \mathbf{K} \exists \mathbf{v} \leq \mathbf{\Phi}(\mathbf{r_x}, \underline{\mathbf{z}}^*)\, \mathbf{A}_\exists(\mathbf{x}, \mathbf{y}, \underline{\mathbf{z}}, \mathbf{v}).$$

For the bounded cases: K. Trans.AMS 2005.

As special case of **general logical metatheorems** due to Gerhardy/K. (Trans. Amer. Math. Soc. 2008) one has:

Corollary (Gerhardy/K., TAMS 2008)

If $\mathcal{A}^\omega[X, d, W]$ proves

$$\forall \mathbf{x} \in \mathbf{P} \forall \mathbf{y} \in \mathbf{K} \forall \mathbf{z} \in \mathbf{X} \forall \mathbf{f} : \mathbf{X} \to \mathbf{X} \big(\mathbf{f} \text{ n.e.} \to \exists \mathbf{v} \in \mathbb{N}\, \mathbf{A}_\exists\big),$$

then one can extract a **computable functional** $\Phi : \mathbb{N}^{\mathbb{N}} \times \mathbb{N} \to \mathbb{N}$ s.t. for all $x \in P, b \in \mathbb{N}$

$$\forall \mathbf{y} \in \mathbf{K} \forall \mathbf{z} \in \mathbf{X} \forall \mathbf{f} : \mathbf{X} \to \mathbf{X} \\ \big(\mathbf{f} \text{ n.e.} \wedge \mathbf{d_X}(\mathbf{z}, \mathbf{f}(\mathbf{z})) \leq \mathbf{b} \to \exists \mathbf{v} \leq \mathbf{\Phi}(\mathbf{r_x}, \mathbf{b}) \mathbf{A}_\exists\big)$$

holds in **all nonempty hyperbolic spaces** (X, d, W).

As special case of **general logical metatheorems** due to Gerhardy/K. (Trans. Amer. Math. Soc. 2008) one has:

Corollary (Gerhardy/K., TAMS 2008)

If $\mathcal{A}^{\omega}[X,d,W]$ proves

$$\forall \mathbf{x} \in \mathbf{P} \forall \mathbf{y} \in \mathbf{K} \forall \mathbf{z} \in \mathbf{X} \forall \mathbf{f} : \mathbf{X} \to \mathbf{X} \big(\mathbf{f} \text{ n.e.} \to \exists \mathbf{v} \in \mathbb{N}\, \mathbf{A}_{\exists}\big),$$

then one can extract a **computable functional** $\Phi : \mathbb{N}^{\mathbb{N}} \times \mathbb{N} \to \mathbb{N}$ s.t. for all $x \in P, b \in \mathbb{N}$

$$\begin{aligned} &\forall \mathbf{y} \in \mathbf{K} \forall \mathbf{z} \in \mathbf{X} \forall \mathbf{f} : \mathbf{X} \to \mathbf{X} \\ &\qquad \big(\mathbf{f} \text{ n.e.} \wedge \mathbf{d_X(z, f(z))} \leq \mathbf{b} \to \exists \mathbf{v} \leq \mathbf{\Phi(r_x, b) A_{\exists}}\big) \end{aligned}$$

holds in **all nonempty hyperbolic spaces** (X, d, W).

Normed case: also $\|z\| \leq b$.

An Example from Ergodic Theory

X **Hilbert space**, $f : X \to X$ **linear** and $\|\mathbf{f(x)}\| \leq \|\mathbf{x}\|$ for all $x \in X$.

$$\mathbf{A_n(x)} := \frac{\mathbf{1}}{\mathbf{n+1}}\mathbf{S_n(x)}, \text{ where } \mathbf{S_n(x)} := \sum_{\mathbf{i=0}}^{\mathbf{n}} \mathbf{f^{(i)}(x)} \quad (\mathbf{n} \geq \mathbf{0})$$

An Example from Ergodic Theory

X **Hilbert space**, $f : X \to X$ **linear** and $\|\mathbf{f(x)}\| \leq \|\mathbf{x}\|$ for all $x \in X$.

$$\mathbf{A_n(x)} := \frac{\mathbf{1}}{\mathbf{n+1}} \mathbf{S_n(x)}, \text{ where } \mathbf{S_n(x)} := \sum_{\mathbf{i=0}}^{\mathbf{n}} \mathbf{f^{(i)}(x)} \quad (\mathbf{n} \geq \mathbf{0})$$

Theorem (von Neumann Mean Ergodic Theorem)

For every $x \in X$, the sequence $(A_n(x))_n$ converges.

An Example from Ergodic Theory

X **Hilbert space**, $f : X \to X$ **linear** and $\|\mathbf{f(x)}\| \leq \|\mathbf{x}\|$ for all $x \in X$.

$$\mathbf{A_n(x)} := \frac{\mathbf{1}}{\mathbf{n+1}}\mathbf{S_n(x)}, \text{ where } \mathbf{S_n(x)} := \sum_{\mathbf{i=0}}^{\mathbf{n}} \mathbf{f^{(i)}(x)} \quad (\mathbf{n} \geq \mathbf{0})$$

Theorem (von Neumann Mean Ergodic Theorem)

For every $x \in X$, the sequence $(A_n(x))_n$ converges.

Avigad/Gerhardy/Towsner (TAMS 2010):

in general **no computable rate of convergence**.

But: **Prim. rec. bound on metastable version!**

An Example from Ergodic Theory

X **Hilbert space**, $f : X \to X$ **linear** and $\|\mathbf{f(x)}\| \leq \|\mathbf{x}\|$ for all $x \in X$.

$$\mathbf{A_n(x) := \frac{1}{n+1} S_n(x), \text{ where } S_n(x) := \sum_{i=0}^{n} f^{(i)}(x) \quad (n \geq 0)}$$

Theorem (von Neumann Mean Ergodic Theorem)

For every $x \in X$, the sequence $(A_n(x))_n$ converges.

Avigad/Gerhardy/Towsner (TAMS 2010):

in general **no computable rate of convergence**.

But: **Prim. rec. bound on metastable version!**

Theorem (Garrett Birkhoff 1939)

Mean Ergodic Theorem holds for uniformly convex Banach spaces.

Since Birkhoff's proof formalizes in $\mathcal{A}^\omega[X, \|\cdot\|, \eta]$ the following is guaranteed:

Since Birkhoff's proof formalizes in $\mathcal{A}^\omega[X, \|\cdot\|, \eta]$ the following is guaranteed:

X uniformly convex Banach space with modulus η and $f : X \to X$ nonexpansive linear operator. Let $b > 0$. Then there is an effective functional Φ in ε, g, b, η s.t. for all $x \in X$ with $\|x\| \leq b$, all $\varepsilon > 0$, all $g : \mathbb{N} \to \mathbb{N}$:

$$\exists \mathbf{n} \leq \boldsymbol{\Phi}(\varepsilon, \mathbf{g}, \mathbf{b}, \eta)\, \forall \mathbf{i}, \mathbf{j} \in [\mathbf{n}, \mathbf{n} + \mathbf{g}(\mathbf{n})] \left(\|\mathbf{A_i(x)} - \mathbf{A_j(x)}\| < \varepsilon\right).$$

Since Birkhoff's proof formalizes in $\mathcal{A}^\omega[X, \|\cdot\|, \eta]$ the following is guaranteed:

X uniformly convex Banach space with modulus η and $f : X \to X$ nonexpansive linear operator. Let $b > 0$. Then there is an effective functional Φ in ε, g, b, η s.t. for all $x \in X$ with $\|x\| \leq b$, all $\varepsilon > 0$, all $g : \mathbb{N} \to \mathbb{N}$:

$$\exists \mathbf{n} \leq \mathbf{\Phi}(\varepsilon, \mathbf{g}, \mathbf{b}, \eta)\, \forall \mathbf{i}, \mathbf{j} \in [\mathbf{n}, \mathbf{n} + \mathbf{g}(\mathbf{n})] \left(\|\mathbf{A_i}(\mathbf{x}) - \mathbf{A_j}(\mathbf{x})\| < \varepsilon\right).$$

Note that $f^* := id$ majorizes f.

Theorem (K./Leuştean, Ergodic Theor. Dynam. Syst. 2009)

X uniformly convex Banach space, η a modulus of uniform convexity and $f : X \to X$ as above, $b > 0$.

Then for all $x \in X$ with $\|x\| \leq b$, all $\varepsilon > 0$, all $g : \mathbb{N} \to \mathbb{N}$:

$$\exists \mathbf{n} \leq \mathbf{\Phi}(\varepsilon, \mathbf{g}, \mathbf{b}, \eta)\, \forall \mathbf{i}, \mathbf{j} \in [\mathbf{n}; \mathbf{n} + \mathbf{g}(\mathbf{n})] \left(\|\mathbf{A}_\mathbf{i}(\mathbf{x}) - \mathbf{A}_\mathbf{j}(\mathbf{x})\| < \varepsilon\right),$$

Theorem (K./Leuştean, Ergodic Theor. Dynam. Syst. 2009)

X uniformly convex Banach space, η a modulus of uniform convexity and $f : X \to X$ as above, $b > 0$.

Then for all $x \in X$ with $\|x\| \le b$, all $\varepsilon > 0$, all $g : \mathbb{N} \to \mathbb{N}$:

$$\exists n \le \Phi(\varepsilon, g, b, \eta)\, \forall i, j \in [n; n + g(n)]\, \left(\|A_i(x) - A_j(x)\| < \varepsilon\right),$$

where

$$\Phi(\varepsilon, g, b, \eta) := M \cdot \tilde{h}^K(0), \text{ with}$$

$$M := \left\lceil \frac{16b}{\varepsilon} \right\rceil, \gamma := \frac{\varepsilon}{16}\eta\left(\frac{\varepsilon}{8b}\right), \quad K := \left\lceil \frac{b}{\gamma} \right\rceil,$$

$$h, \tilde{h} : \mathbb{N} \to \mathbb{N},\ h(n) := 2(Mn + g(Mn)), \quad \tilde{h}(n) := \max_{i \le n} h(i).$$

Theorem (K./Leuştean, Ergodic Theor. Dynam. Syst. 2009)

X uniformly convex Banach space, η a modulus of uniform convexity and $f : X \to X$ as above, $b > 0$.
Then for all $x \in X$ with $\|x\| \leq b$, all $\varepsilon > 0$, all $g : \mathbb{N} \to \mathbb{N}$:

$$\exists n \leq \Phi(\varepsilon, g, b, \eta)\, \forall i, j \in [n; n + g(n)] \left(\|A_i(x) - A_j(x)\| < \varepsilon\right),$$

where

$$\Phi(\varepsilon, g, b, \eta) := M \cdot \tilde{h}^K(0), \text{ with}$$
$$M := \left\lceil \frac{16b}{\varepsilon} \right\rceil, \gamma := \frac{\varepsilon}{16}\eta\left(\frac{\varepsilon}{8b}\right), \quad K := \left\lceil \frac{b}{\gamma} \right\rceil,$$
$$h, \tilde{h} : \mathbb{N} \to \mathbb{N},\ h(n) := 2(Mn + g(Mn)), \quad \tilde{h}(n) := \max_{i \leq n} h(i).$$

Special Hilbert case: treated prior by Avigad/Gerhardy/Towsner (again based on logical metatheorem).

Bounding the number of fluctuations

We say that (x_n) admits k ε-fluctuations if there are $i_1 \leq j_1 \leq \ldots i_k \leq j_k$ s.t. $\|x_{j_n} - x_{i_n}\| \geq \varepsilon$ for $n = 1, \ldots, k$.

Bounding the number of fluctuations

We say that (x_n) admits k ε-fluctuations if there are $i_1 \leq j_1 \leq \ldots i_k \leq j_k$ s.t. $\|x_{j_n} - x_{i_n}\| \geq \varepsilon$ for $n = 1, \ldots, k$.

As a corollary to our analysis of Birkhoff's proof, Avigad and Rute showed

Theorem (Avigad, Rute (2012))

$(A_n(x))$ admits at most

$$2\log(\mathbf{M}) \cdot \frac{\mathbf{b}}{\varepsilon} + \frac{\mathbf{b}}{\gamma} \cdot (2\log(2\mathbf{M}) \cdot \frac{\mathbf{b}}{\varepsilon} + \frac{\mathbf{b}}{\gamma}$$

many fluctuations.

Bounding the number of fluctuations

We say that (x_n) admits k ε-fluctuations if there are $i_1 \leq j_1 \leq \dots i_k \leq j_k$ s.t. $\|x_{j_n} - x_{i_n}\| \geq \varepsilon$ for $n = 1, \dots, k$.

As a corollary to our analysis of Birkhoff's proof, Avigad and Rute showed

Theorem (Avigad, Rute (2012))

$(A_n(x))$ admits at most

$$2\log(M) \cdot \frac{b}{\varepsilon} + \frac{b}{\gamma} \cdot (2\log(2M) \cdot \frac{b}{\varepsilon} + \frac{b}{\gamma}$$

many fluctuations.

For the Hilbert space case: first fluctuation bounds by Jones, Ostrovskii, Rosenblatt 1996.

Tao also established (without bound) a uniform version (in a special case) of the Mean Ergodic Theorem as base step for a generalization to commuting families of operators.

Tao also established (without bound) a uniform version (in a special case) of the Mean Ergodic Theorem as base step for a generalization to commuting families of operators.

‘We shall establish Theorem 1.6 by “finitary ergodic theory” techniques, reminiscent of those used in [Green-Tao]...’ ‘The main advantage of working in the finitary setting ... is that the underlying dynamical system becomes extremely explicit’...‘In proof theory, this finitisation is known as Gödel functional interpretation...which is also closely related to the Kreisel no-counterexample interpretation’
(T. Tao: Norm convergence of multiple ergodic averages for commuting transformations, Ergodic Theor. and Dynam. Syst. 28, 2008)

Literature

1) Bridges, D.S., Recent progress in constructive approximation theory. In: Troelstra, A.S./van Dalen, D. (eds.) The L.E.J. Brouwer Centenary Symposium. North-Holland, Amsterdam, pp. 41-50 (1982).

2) Briseid, E.M., Fixed points of generalized contractive mappings. Journal of Nonlinear and Convex Analysis **9**, pp. 181-204 (2008).

3) Briseid, E.M., Logical aspects of rates of convergence in metric spaces. J. Symb. Log. **74**, pp. 1401-1428 (2009.)

4) Ko, K.-I., On the computational complexity of best Chebycheff approximation. J. of Complexity **2**, pp. 95-120 (1986).

5) Kohlenbach, U., Effective moduli from ineffective uniqueness proofs. An unwinding of de La Vallee Poussin's proof for Chebycheff approximation. Annals of Pure and Applied Logic 64, pp. 27-94 (1993).

6) Kohlenbach, U., New effective moduli of uniqueness and uniform a-priori estimates for constants of strong unicity by logical analysis of known proofs in best approximation theory. Numer. Funct. Anal. and Optimiz. **14**, pp. 581-606 (1993).

7) Kohlenbach, U., Applied Proof Theory. Springer Monograph in Mathematics. xx+536 pp., 2008.

8) Kohlenbach, U., Analysing proofs in analysis. In: W. Hodges, M. Hyland, C. Steinhorn, J. Truss, editors, *Logic: from Foundations to Applications. European Logic Colloquium* (Keele, 1993), pp. 225–260, Oxford University Press (1996).

9) Kohlenbach, U., Oliva, P., Proof mining in L_1-approximation. Ann. Pure Appl. Logic **121**, pp. 1-38 (2003).

10) Oliva, P., On the computational complexity of best L_1-Approximation. Math. Logic. Quart. **48, suppl. I**, pp. 66-77 (2002).

Literature

11) Avigad, J., Iovino, J., Ultraproducts and metastability. New York J. of Math. **19**, pp. 713-727 (2013).

12) Avigad, J., Rute, J., Oscillation and the mean ergodic theorem for uniformly convex Banach spaces. Ergod. Th. & Dynam. Sys. **35**, pp. 1009-1027 (2015).

13) Gerhardy, P., Kohlenbach, U., General logical metatheorems for functional analysis, Trans. Amer. Math. Soc. **360** , pp. 2615-2660 (2008).

14) Günzel, D., Kohlenbach, U., Logical metatheorems for abstract spaces axiomatized in positive bounded logic. Advances in Mathematics **290**, pp. 503-551 (2016).

15) Kohlenbach, U., Some logical metatheorems with applications in functional analysis. Trans. Amer. Math. Soc. **357**, pp. 89-128 (2005).

16) Kohlenbach, U., Applied Proof Theory. Springer Monograph in Mathematics. xx+536pp., 2008.

17) Kohlenbach, U., On quantitative versions of theorems due to F.E. Browder and R. Wittmann, Advances in Mathematics **226**, pp. 2764-2795 (2011).

18) Kohlenbach, U., A uniform quantitative form of sequential weak compactness and Baillon's nonlinear ergodic theorem. Communications in Contemporary Mathematics 14, 20pp. (2012).

19) Kohlenbach, U., Leuştean, L., On the computational content of convergence proofs via Banach limits. Phil. Trans. Royal Soc. A **370**, pp. 3449-3463 (2012).

20) Kohlenbach, U., Leuştean, L., Effective metastability of Halpern iterates in CAT(0) spaces. Advances in Mathematics vol. 231, pp. 2526-2556 (2012). Addendum: Advances in Mathematics vol. 250, pp. 650-651, 2014.

21) Leuştean, L., Nicolae, A., Effective results on nonlinear ergodic averages in CAT(k) spaces. Ergod. Th. & Dynam. Sys., DOI: 10.1017/etds.2015.31, 2015.

22) Safarik, P., A quantitative strong linear ergodic theorem for Hilbert spaces. J. Math. Anal. Appl. **391**, pp. 26-37 (2012).

23) Schade, K., Kohlenbach, U., Effective metastability for modified Halpern iterations in CAT(0) spaces. Fixed Point Theory and Applications 2012:191, 19pp.

24) Wittmann, R., Approximation of fixed points of nonexpansive mappings. Arch. Math. **58**, pp. 486-491 (1992).

Lecture IV

Applications to fixed point theory and convex optimization

General context:

- (X, d, W) is a (non-empty) **hyperbolic space**.

Applications to fixed point theory and convex optimization

General context:

- (X, d, W) is a (non-empty) **hyperbolic space**.
- $f : X \to X$ is a **nonexpansive mapping**.

Applications to fixed point theory and convex optimization

General context:

- (X, d, W) is a (non-empty) **hyperbolic space**.
- $f : X \to X$ is a **nonexpansive mapping**.
- (λ_n) is a sequence in $[0, 1]$ that is **bounded away from** 1 and **divergent in sum**.

Applications to fixed point theory and convex optimization

General context:

- (X, d, W) is a (non-empty) **hyperbolic space**.
- $f : X \to X$ is a **nonexpansive mapping**.
- (λ_n) is a sequence in $[0, 1]$ that is **bounded away from** 1 and **divergent in sum**.
- $x_{n+1} = (1 - \lambda_n)x_n \oplus \lambda_n f(x_n)$ (**Krasnoselski-Mann iter.**).

Applications to fixed point theory and convex optimization

General context:

- (X, d, W) is a (non-empty) **hyperbolic space**.
- $f : X \to X$ is a **nonexpansive mapping**.
- (λ_n) is a sequence in $[0, 1]$ that is **bounded away from 1** and **divergent in sum**.
- $x_{n+1} = (1 - \lambda_n)x_n \oplus \lambda_n f(x_n)$ (**Krasnoselski-Mann iter.**).

Theorem (Ishikawa 1976, Goebel/Kirk 1983)

(Ishikawa I)

$$\text{If } (x_n) \text{ is bounded, then } d(x_n, f(x_n)) \to 0.$$

Applications to fixed point theory and convex optimization

General context:

- (X, d, W) is a (non-empty) **hyperbolic space**.
- $f : X \to X$ is a **nonexpansive mapping**.
- (λ_n) is a sequence in $[0, 1]$ that is **bounded away from** 1 and **divergent in sum**.
- $x_{n+1} = (1 - \lambda_n)x_n \oplus \lambda_n f(x_n)$ (**Krasnoselski-Mann iter.**).

Theorem (Ishikawa 1976, Goebel/Kirk 1983)

(Ishikawa I)

If (x_n) is bounded, then $d(x_n, f(x_n)) \to 0$.

Crucial: $(d(x_n, f(x_n)))_n$ is nonincreasing!.

Logical analysis of the proof of Ishikawa's theorem

Let $K \in \mathbb{N}$ and $\alpha : \mathbb{N} \to \mathbb{N}$ be such that

$$(\lambda_n)_{n\in\mathbb{N}} \in [0, 1 - \frac{1}{K}]^{\mathbb{N}} \text{ and } \forall n \in \mathbb{N}(n \leq \sum_{i=0}^{\alpha(n)} \lambda_i).$$

Logical metatheorem applied to proof of Ishikawa's theorem yields computable Ψ, Φ s.t. for all $k \in \mathbb{N}$ and n.e. f

$$\forall i, j \leq \Psi(K, \alpha, b, \tilde{b}, k) \left(d(x, f(x)) \leq b \wedge d(x_i, x_j) \leq \tilde{b}\right) \to \forall m \geq \Phi(K, \alpha, b, \tilde{b}, k) \left(d(x_m, f(x_m)) < 2^{-k}\right).$$

holds in **any (nonempty) hyperbolic space** (X, d, W).

Theorem (K.2007, K./Leustean AAA 2003)

$(X,d,W),(\lambda_n),K,\alpha$ as above, $f : X \to X$ nonexpansive the following holds for all $\varepsilon, b, \tilde{b} > 0$:

> If $\mathbf{d(x,f(x)) \leq b}$ and $\forall \mathbf{i} \leq \mathbf{\Phi} \forall \mathbf{j} \leq \alpha(\mathbf{\Phi}, \mathbf{M})\ (\mathbf{d(x_i, x_{i+j}) \leq \tilde{b}})$ then $\forall \mathbf{n} \geq \mathbf{\Phi}\ \big(\mathbf{d(x_n, f(x_n)) \leq \varepsilon}\big)$,

Theorem (K.2007, K./Leustean AAA 2003)

$(X, d, W), (\lambda_n), K, \alpha$ as above, $f : X \to X$ nonexpansive the following holds for all $\varepsilon, b, \tilde{b} > 0$:

$$\text{If } \mathbf{d(x, f(x)) \leq b} \text{ and } \forall \mathbf{i} \leq \mathbf{\Phi} \forall \mathbf{j} \leq \alpha(\mathbf{\Phi}, \mathbf{M})\ (\mathbf{d(x_i, x_{i+j}) \leq \tilde{b}})$$
$$\text{then } \forall \mathbf{n} \geq \mathbf{\Phi}\ \big(\mathbf{d(x_n, f(x_n)) \leq \varepsilon}\big),$$

where

$$\mathbf{\Phi} := \mathbf{\Phi}(\mathbf{K}, \alpha, \mathbf{b}, \tilde{\mathbf{b}}, \varepsilon) := \widehat{\alpha}\left(\left\lceil \frac{\tilde{\mathbf{b}} \cdot \exp\left(\mathbf{K} \cdot \left(\frac{\tilde{\mathbf{b}}+3\mathbf{b}}{\varepsilon}+1\right)\right)}{\varepsilon} \right\rceil \dot{-} 1, \mathbf{M}\right),$$

$$\mathbf{M} := \left\lceil \frac{\tilde{\mathbf{b}}+3\mathbf{b}}{\varepsilon} \right\rceil,$$

$$\widehat{\alpha}(\mathbf{0}, \mathbf{n}) := \tilde{\alpha}(\mathbf{0}, \mathbf{n}),\ \widehat{\alpha}(\mathbf{i}+\mathbf{1}, \mathbf{n}) := \tilde{\alpha}(\widehat{\alpha}(\mathbf{i}, \mathbf{n}), \mathbf{n}) \text{ with}$$

$$\tilde{\alpha}(\mathbf{i}, \mathbf{n}) := \mathbf{i} + \alpha(\mathbf{i}, \mathbf{n}) \quad (\mathbf{i}, \mathbf{n} \in \mathbb{N})$$

Theorem (K.2007, K./Leustean AAA 2003)

$(X,d,W),(\lambda_n),K,\alpha$ as above, $f:X\to X$ nonexpansive the following holds for all $\varepsilon, b, \tilde{b} > 0$:

$$\text{If } \mathbf{d(x,f(x))\le b} \text{ and } \forall \mathbf{i} \le \mathbf{\Phi}\forall \mathbf{j}\le \alpha(\mathbf{\Phi},\mathbf{M})\ (\mathbf{d(x_i,x_{i+j})}\le \tilde{\mathbf{b}})$$
$$\text{then } \forall \mathbf{n}\ge \mathbf{\Phi}\ \big(\mathbf{d(x_n,f(x_n))}\le \varepsilon\big),$$

where

$$\mathbf{\Phi} := \mathbf{\Phi}(\mathbf{K},\alpha,\mathbf{b},\tilde{\mathbf{b}},\varepsilon) := \widehat{\alpha}\left(\left\lceil \frac{\tilde{\mathbf{b}}\cdot\exp\left(\mathbf{K}\cdot\left(\frac{\tilde{\mathbf{b}}+3\mathbf{b}}{\varepsilon}+1\right)\right)}{\varepsilon}\right\rceil \dot{-}1,\mathbf{M}\right),$$

$$\mathbf{M} := \left\lceil \frac{\tilde{\mathbf{b}}+3\mathbf{b}}{\varepsilon}\right\rceil,$$

$$\widehat{\alpha}(\mathbf{0},\mathbf{n}) := \tilde{\alpha}(\mathbf{0},\mathbf{n}),\ \widehat{\alpha}(\mathbf{i}+\mathbf{1},\mathbf{n}) := \tilde{\alpha}(\widehat{\alpha}(\mathbf{i},\mathbf{n}),\mathbf{n}) \text{ with}$$

$$\tilde{\alpha}(\mathbf{i},\mathbf{n}) := \mathbf{i}+\alpha(\mathbf{i},\mathbf{n})\quad (\mathbf{i},\mathbf{n}\in\mathbb{N})$$

with $\boldsymbol{\alpha}$ s.t.

$$\forall \mathbf{i},\mathbf{n}\in\mathbb{N}\Big((\alpha(\mathbf{i},\mathbf{n})\le \alpha(\mathbf{i}+\mathbf{1},\mathbf{n}))\wedge\Big(\mathbf{n}\le \sum_{\mathbf{s}=\mathbf{i}}^{\mathbf{i}+\alpha(\mathbf{i},\mathbf{n})-\mathbf{1}} \lambda_{\mathbf{s}}\Big)\Big).$$

Known uniformity results in the bounded case

blue = hyperbolic, **green** = dir.nonex., **red** = both.

- Krasnoselski(1955):Xunif.convex,C compact,λ_k=$\frac{1}{2}$,no uniform.
- Browder/Petryshyn(1967):Xunif.convex,$\lambda_k = \lambda$, no uniformity.
- Groetsch(1972): X unif. convex, general λ_k, X, no uniformity
- Ishikawa (1976): No uniformity
- Edelstein/O'Brien (1978): Uniformity w.r.t. $x_0 \in C$ ($\lambda_k := \lambda$)
- Goebel/Kirk (1982): Uniformity w.r.t. x_0 **and** f. General λ_k
- Kirk/Martinez (1990): Uniformity for unif. convex $X, \lambda := 1/2$
- Goebel/Kirk (1990): Conjecture: no uniformity w.r.t. C
- Baillon/Bruck (1996): Uniformity w.r.t. x_0, f, C for $\lambda_k := \lambda$
- **Kirk (2001): Uniformity w.r.t. x_0, f for constant λ**
- Kohlenbach (2001): Full uniformity for general λ_k
- **K./Leustean (2003): Full uniformity for general λ_k**

Corollary (K.2008)

Let $\lambda_n := \lambda \in (0, 1.)$

$$\textit{If } \lim_{n\to\infty} \frac{c(n)}{n} \to 0, \ \textit{where } c(n) := \max\{d(x, x_j) : j \leq n\},$$

then

$$\lim_{n\to\infty} d(x_n, f(x_n)) = 0.$$

Corollary (K.2008)

Let $\lambda_n := \lambda \in (0, 1.)$

$$\textit{If } \lim_{n\to\infty} \frac{c(n)}{n} \to 0, \ \textit{where } c(n) := \max\{d(x, x_j) : j \leq n\},$$

then

$$\lim_{n\to\infty} d(x_n, f(x_n)) = 0.$$

Result optimal: $c(n) \leq K \cdot n$ not sufficient!

Theorem (Ishikawa, Goebel, Kirk)

(Ishikawa II) If previous assumptions and X **compact**, then (x_n) converges towards a fixed point.

Theorem (Ishikawa, Goebel, Kirk)

(Ishikawa II) If previous assumptions and X **compact**, then (x_n) converges towards a fixed point.

Proof: Since X is compact, (x_n) possesses a **convergent subsequence** (x_{n_k}). Let $\widehat{x} := \lim x_{n_k}$. Since by Ishikawa I, (x_n) (and hence x_{n_k}) is an asymptotic fixed point sequence and f is continuous, $\widehat{x}$ is a fixed point of f. The claim now follows from the following easy inequality

$$\forall u \in Fix(f) \forall n \in \mathbb{N} \, (d(x_{n+1}, u) \leq d(x_n, u)).$$

Theorem (Ishikawa, Goebel, Kirk)

(Ishikawa II) If previous assumptions and X **compact**, then (x_n) converges towards a fixed point.

Proof: Since X is compact, (x_n) possesses a **convergent subsequence** (x_{n_k}). Let $\widehat{x} := \lim x_{n_k}$. Since by Ishikawa I, (x_n) (and hence x_{n_k}) is an asymptotic fixed point sequence and f is continuous, $\widehat{x}$ is a fixed point of f. The claim now follows from the following easy inequality

$$\forall \mathbf{u} \in \mathbf{Fix(f)} \forall \mathbf{n} \in \mathbb{N} \, (\mathbf{d(x_{n+1}, u)} \leq \mathbf{d(x_n, u)}).$$

Problem: No computable rate of convergence.

Cauchy property $\forall\exists\forall$ rather than $\forall\exists$ (asymptotic regularity).

Theorem (Ishikawa, Goebel, Kirk)

(Ishikawa II) If previous assumptions and X **compact**, then (x_n) converges towards a fixed point.

Proof: Since X is compact, (x_n) possesses a **convergent subsequence** (x_{n_k}). Let $\widehat{x} := \lim x_{n_k}$. Since by Ishikawa I, (x_n) (and hence x_{n_k}) is an asymptotic fixed point sequence and f is continuous, $\widehat{x}$ is a fixed point of f. The claim now follows from the following easy inequality

$$\forall \mathbf{u} \in \mathbf{Fix(f)} \forall \mathbf{n} \in \mathbb{N}\, (\mathbf{d}(\mathbf{x_{n+1}}, \mathbf{u}) \leq \mathbf{d}(\mathbf{x_n}, \mathbf{u})).$$

Problem: No computable rate of convergence.
Cauchy property $\forall\exists\forall$ rather than $\forall\exists$ (asymptotic regularity).
Best possible: Bound on the **no-counterexample interpretation**:

$$\text{(H) } \forall \mathbf{g} : \mathbb{N} \to \mathbb{N} \forall \mathbf{k} \exists \mathbf{n} \forall \mathbf{j_1}, \mathbf{j_2} \in [\mathbf{n}; \mathbf{n} + \mathbf{g}(\mathbf{n})](\mathbf{d}(\mathbf{x_{j_1}}, \mathbf{x_{j_2}}) < \mathbf{2^{-k}}).$$

Logical Metatheorem for Compact Spaces

We add to $\mathcal{T}[X, d, W]$ compactness via

- A constant $\gamma : \mathbb{N} \to \mathbb{N}$ with an axiom expressing that γ is a **modulus of total boundedness:**

$$(\mathrm{TOT}): \ \forall \mathbf{k} \in \mathbb{N}, \mathbf{x}_{(\cdot)}^{\mathbb{N}\to \mathbf{X}} \exists \mathbf{i}, \mathbf{j}\, (\mathbf{i} < \mathbf{j} \leq \gamma(\mathbf{k}) \ \wedge \ \mathbf{d}(\mathbf{x_i}, \mathbf{x_j}) \leq \mathbf{2^{-k}})$$

Logical Metatheorem for Compact Spaces

We add to $\mathcal{T}[X, d, W]$ compactness via

- A constant $\gamma : \mathbb{N} \to \mathbb{N}$ with an axiom expressing that γ is a **modulus of total boundedness:**

$$(\mathrm{TOT}): \ \forall \mathbf{k} \in \mathbb{N}, \mathbf{x}_{(\cdot)}^{\mathbb{N}\to \mathbf{X}} \exists \mathbf{i}, \mathbf{j}\, (\mathbf{i} < \mathbf{j} \leq \gamma(\mathbf{k}) \ \wedge \ \mathbf{d}(\mathbf{x_i}, \mathbf{x_j}) \leq \mathbf{2}^{-\mathbf{k}})$$

- An axiom $\mathcal{C}$ expressing **completeness via an operator** C that maps Cauchy sequences to their limit.

Logical Metatheorem for Compact Spaces

We add to $\mathcal{T}[X, d, W]$ compactness via

- A constant $\gamma : \mathbb{N} \to \mathbb{N}$ with an axiom expressing that γ is a **modulus of total boundedness:**

$$(\mathrm{TOT}): \ \forall \mathbf{k} \in \mathbb{N}, \mathbf{x}_{(\cdot)}^{\mathbb{N}\to\mathbf{X}} \exists \mathbf{i}, \mathbf{j} \, (\mathbf{i} < \mathbf{j} \leq \gamma(\mathbf{k}) \ \wedge \ \mathbf{d}(\mathbf{x_i}, \mathbf{x_j}) \leq \mathbf{2^{-k}})$$

- An axiom $\mathcal{C}$ expressing **completeness via an operator** C that maps Cauchy sequences to their limit.

The completeness issue is of minor relevance for the case at hand, but the total boundedness is.

Corresponding theory: $\mathcal{T}[X, d, W, \mathcal{C}, \mathrm{TOT}]$.

Guaranteed by logical metatheorem

From the fact that the proof of

$$\text{Ishikawa I}(x_n) \ \wedge \ \text{BW}(x_n) \rightarrow \ \text{Ishikawa II}(x_n)$$

can be formalized in an appropriate fragment of $\mathcal{A}^\omega[X, d, W, \mathcal{C}, \text{TOT}]$ it follows:

Guaranteed by logical metatheorem

From the fact that the proof of

$$\text{Ishikawa I}(x_n) \ \wedge \ \text{BW}(x_n) \rightarrow \ \text{Ishikawa II}(x_n)$$

can be formalized in an appropriate fragment of $\mathcal{A}^{\omega}[X, d, W, \mathcal{C}, \text{TOT}]$ it follows:

Theorem

There exists a **primitive recursive functional Ψ** such that for any **rate of asymptotic regularity Φ** and any **modulus of total boundedness γ** for C, any g, k :

$$\exists n \leq \Psi(\Phi, \gamma, g, k) \forall j_1, j_2 \in [n; n + g(n)](d(x_{j_1}, x_{j_2}) < 2^{-k}).$$

Theorem (K., Nonlinear Analysis 2005)

A bound satisfying the previous theorem is given by

$$\Psi(\Phi, \gamma, g, k) := \max_{i \le \gamma(k)} \Psi_0(i, k, g, \Phi),$$

where

$$\begin{cases} \Psi_0(0, k, g, \Phi) := 0 \\ \Psi_0(n+1, k, g, \Phi) := \Phi\left(2^{-k-2}/(\max_{i \le n} g(\Psi_0(i, k, g, \Phi)) + 1)\right). \end{cases}$$

Kirk's theorem for asymptotic contractions

Definition (Kirk JMAA03)

(X, d) metric space. $f : X \to X$ is an **asymptotic contraction** with moduli $\Phi, \Phi_n : [0, \infty) \to [0, \infty)$ if Φ, Φ_n are continuous, $\Phi(s) < s$ for all $s > 0$ and

$$\forall n \in \mathbb{N} \forall x, y \in X (d(f^n(x), f^n(y)) \le \Phi_n(d(x, y)),$$

and $\Phi_n \to \Phi$ uniformly on the range of d.

Kirk's theorem for asymptotic contractions

Definition (Kirk JMAA03)

(X, d) metric space. $f : X \to X$ is an **asymptotic contraction** with moduli $\Phi, \Phi_n : [0, \infty) \to [0, \infty)$ if Φ, Φ_n are continuous, $\Phi(s) < s$ for all $s > 0$ and

$$\forall n \in \mathbb{N} \forall x, y \in X (d(f^n(x), f^n(y)) \leq \Phi_n(d(x, y)),$$

and $\Phi_n \to \Phi$ uniformly on the range of d.

Theorem (Kirk JMAA03)

(X, d) complete metric space, $f : X \to X$ continuous asymptotic contraction with some orbit bounded. Then f has a unique fixed point $p \in X$ and $(f^n(x_0))$ converges to p for each $x_0 \in X$.

Kirk's theorem for asymptotic contractions

Definition (Kirk JMAA03)

(X, d) metric space. $f : X \to X$ is an **asymptotic contraction** with moduli $\Phi, \Phi_n : [0, \infty) \to [0, \infty)$ if Φ, Φ_n are continuous, $\Phi(s) < s$ for all $s > 0$ and

$$\forall n \in \mathbb{N} \forall x, y \in X(d(f^n(x), f^n(y)) \leq \Phi_n(d(x, y)),$$

and $\Phi_n \to \Phi$ uniformly on the range of d.

Theorem (Kirk JMAA03)

(X, d) complete metric space, $f : X \to X$ continuous asymptotic contraction with some orbit bounded. Then f has a unique fixed point $p \in X$ and $(f^n(x_0))$ converges to p for each $x_0 \in X$.

(Proof uses ultrapower structures!)

- By proof mining P. Gerhardy (JMAA 2006, communicated by Kirk) obtained an **effective rate of proximity** Φ in appropriate moduli with elementary proof such that for the fixed point p

$$\forall \varepsilon > 0 \exists \mathbf{n} \leq \mathbf{\Phi}(\varepsilon)\ (\mathbf{d}(\mathbf{p}, \mathbf{f}^{\mathbf{n}}(\mathbf{x}_0)) < \varepsilon).$$

- By proof mining P. Gerhardy (JMAA 2006, communicated by Kirk) obtained an **effective rate of proximity** Φ in appropriate moduli with elementary proof such that for the fixed point p

$$\forall \varepsilon > 0 \exists n \leq \Phi(\varepsilon)\ (d(p, f^n(x_0)) < \varepsilon).$$

- Using the uniformity of Gerhardy's result, E.M.Briseid (JMAA 2007) constructed an effective **full rate of convergence.**

- By proof mining P. Gerhardy (JMAA 2006, communicated by Kirk) obtained an **effective rate of proximity** Φ in appropriate moduli with elementary proof such that for the fixed point p

$$\forall \varepsilon > 0 \exists \mathbf{n} \leq \mathbf{\Phi}(\varepsilon)\ (\mathbf{d}(\mathbf{p}, \mathbf{f}^{\mathbf{n}}(\mathbf{x}_0)) < \varepsilon).$$

- Using the uniformity of Gerhardy's result, E.M.Briseid (JMAA 2007) constructed an effective **full rate of convergence.**
- As a consequence of his analysis E.M.Briseid showed that the $(f^n(x_0))$ **is redundant** to assume: rate of convergence using only $b \geq d(x, f(x))$ (Fixed Point Theory 2007, Int. J. Math. Stat. 2010).

- By proof mining P. Gerhardy (JMAA 2006, communicated by Kirk) obtained an **effective rate of proximity** Φ in appropriate moduli with elementary proof such that for the fixed point p

$$\forall \varepsilon > 0 \exists n \leq \Phi(\varepsilon)\ (d(p, f^n(x_0)) < \varepsilon).$$

- Using the uniformity of Gerhardy's result, E.M.Briseid (JMAA 2007) constructed an effective **full rate of convergence.**
- As a consequence of his analysis E.M.Briseid showed that the $(f^n(x_0))$ **is redundant** to assume: rate of convergence using only $b \geq d(x, f(x))$ (Fixed Point Theory 2007, Int. J. Math. Stat. 2010).
- E.M.Briseid showed that for bounded metric spaces the existence of a x_0-uniform rate of convergence **implies** that f is asymptotically contractive (JMAA 2007). Also: new uniformity results generalizing Reich et al (2007).

Asymptotic regularity of pseudocontractions

Let X be a Banach space, $C \subset X$ a bounded convex subset and $f : C \to C$ a Lipschitzian pseudocontraction, i.e.

Asymptotic regularity of pseudocontractions

Let X be a Banach space, $C \subset X$ a bounded convex subset and $f : C \to C$ a Lipschitzian pseudocontraction, i.e.

$$\forall \mathbf{u}, \mathbf{v} \in \mathbf{C} \forall \boldsymbol{\lambda} > 1 \left((\boldsymbol{\lambda} - 1)\|\mathbf{u} - \mathbf{v}\| \leq \|(\boldsymbol{\lambda} \mathbf{I} - \mathbf{f})(\mathbf{u}) - (\boldsymbol{\lambda} \mathbf{I} - \mathbf{f})(\mathbf{v})\| \right).$$

Asymptotic regularity of pseudocontractions

Let X be a Banach space, $C \subset X$ a bounded convex subset and $f : C \to C$ a Lipschitzian pseudocontraction, i.e.

$$\forall \mathbf{u}, \mathbf{v} \in \mathbf{C} \forall \boldsymbol{\lambda} > 1 \left((\boldsymbol{\lambda} - 1)\|\mathbf{u} - \mathbf{v}\| \leq \|(\boldsymbol{\lambda}\mathbf{I} - \mathbf{f})(\mathbf{u}) - (\boldsymbol{\lambda}\mathbf{I} - \mathbf{f})(\mathbf{v})\| \right).$$

Let

$$\mathbf{x_n + 1 := (1 - \boldsymbol{\lambda}_n)x_n + \boldsymbol{\lambda}_n f(x_n) - \boldsymbol{\lambda}_n \boldsymbol{\theta}_n (x_n - x_1)},$$

where $(\lambda_n), (\theta_n) \subset (0, 1]$ with

Asymptotic regularity of pseudocontractions

Let X be a Banach space, $C \subset X$ a bounded convex subset and $f : C \to C$ a Lipschitzian pseudocontraction, i.e.

$$\forall \mathbf{u}, \mathbf{v} \in \mathbf{C} \forall \boldsymbol{\lambda} > 1 \left((\boldsymbol{\lambda} - 1)\|\mathbf{u} - \mathbf{v}\| \leq \|(\boldsymbol{\lambda} \mathbf{I} - \mathbf{f})(\mathbf{u}) - (\boldsymbol{\lambda} \mathbf{I} - \mathbf{f})(\mathbf{v})\| \right).$$

Let

$$\mathbf{x_n + 1 := (1 - \lambda_n)x_n + \lambda_n f(x_n) - \lambda_n \theta_n (x_n - x_1)},$$

where $(\lambda_n), (\theta_n) \subset (0, 1]$ with

$$\textbf{(i)}\ \lim \theta_n = 0,\ \textbf{(ii)}\ \sum_{n=1}^{\infty} \lambda_n \theta_n = \infty,\ \ \textbf{(iii)}\ \lim \frac{\lambda_n}{\theta_n} = 0,$$

$$\textbf{(iv)}\ \lim \frac{\frac{\theta_{n-1}}{\theta_n} - 1}{\lambda_n \theta_n} = 0,\ \ \textbf{(v)}\ \lambda_n (1 + \theta_n) \leq 1.$$

Convergence of Bruck's formula for Lipschitzian pseudocontractions

Theorem (Chidume,Zegeye 2004): $\lim_{n\to\infty} \|x_n - f(x_n)\| = 0.$

Convergence of Bruck's formula for Lipschitzian pseudocontractions

Theorem (Chidume,Zegeye 2004): $\lim_{n\to\infty} \|x_n - f(x_n)\| = 0.$

Let $M \geq diam(C)$ and $(\lambda_n), (\theta_n) \subset (0,1]$ with rates of conv./div.
$R_i : (0,\infty) \to \mathbb{N}$

1. $\forall \varepsilon > 0 \forall n \geq R_1(\varepsilon) (\theta_n \leq \varepsilon)$,
2. $\forall x \in (0,\infty) \left(\sum_{n=1}^{R_2(x)} \lambda_n \theta_n \geq x \right)$,
3. $\forall \varepsilon > 0 \forall n \geq R_3(\varepsilon) (\lambda_n \leq \theta_n \varepsilon)$,
4. $\forall \varepsilon > 0 \forall n \geq R_4(\varepsilon) \left(\frac{\left| \frac{\theta_{n-1}}{\theta_n} - 1 \right|}{\lambda_n \theta_n} \leq \varepsilon \right)$.

Rate of convergence extracted from Chidume/Zegeye (2004):

Rate of convergence extracted from Chidume/Zegeye (2004):

Theorem (D. Körnlein/K. Nonlinear Analysis 2011)

$$\forall \varepsilon > 0 \forall n \geq \Psi(M, L, R_1, R_2, R_3, R_4, \varepsilon) (\|x_n - fx_n\| < \varepsilon)$$

Rate of convergence extracted from Chidume/Zegeye (2004):

Theorem (D. Körnlein/K. Nonlinear Analysis 2011)

$$\forall \varepsilon > 0 \forall n \geq \Psi(M, L, R_1, R_2, R_3, R_4, \varepsilon) (\|x_n - fx_n\| < \varepsilon)$$

where

$$\Psi(M, L, R_1, R_2, R_3, R_4, \varepsilon) = \max\left\{N_2(C) + 1, R_1\left(\frac{\varepsilon}{2M}\right) + 1\right\}$$

and

$$N_1(\varepsilon) := \max\left\{R_3\left(\frac{\varepsilon}{4M^2(2+L)}\right), R_4\left(\sqrt{\frac{\varepsilon}{M^2}+1} - 1\right)\right\},$$

$$N_2(x) := R_2\left(\frac{x}{2}\right) + 1,$$

$$C := \frac{8(1+L)^2 M^2}{\varepsilon^2} + 2\left(N_1\left(\frac{\varepsilon^2}{8(1+L)^2}\right) - 1\right).$$

A quantitative image recovery theorem

Let $\mathbf{H}$ be a Hilbert space, $\mathbf{P_i : H \to C_i}$ metric projection onto the closed and convex $\mathbf{C_i \subseteq H}$.

A quantitative image recovery theorem

Let $\mathbf{H}$ be a Hilbert space, $\mathbf{P_i : H \to C_i}$ metric projection onto the closed and convex $\mathbf{C_i} \subseteq \mathbf{H}$.

Image recovery problem: find a $\mathbf{p} \in \mathbf{C_0 := \cap_{1 \leq i \leq r} C_i}$.

A quantitative image recovery theorem

Let $\mathbf{H}$ be a Hilbert space, $\mathbf{P_i : H \to C_i}$ metric projection onto the closed and convex $\mathbf{C_i} \subseteq \mathbf{H}$.

Image recovery problem: find a $\mathbf{p} \in \mathbf{C_0} := \cap_{1 \le i \le r} \mathbf{C_i}$.

Define $\mathbf{T_i} := \mathbf{I} + \lambda_i(\mathbf{P_i} - \mathbf{I})$ for $0 < \lambda_i \le 2, \lambda_1 < 2$ and $\mathbf{T} := \sum\limits_{i=1}^{r} \alpha_i \mathbf{T_i}$, where $\alpha_i \in (0,1), \ \sum \alpha_i = 1$.

A quantitative image recovery theorem

Let $\mathbf{H}$ be a Hilbert space, $\mathbf{P_i : H \to C_i}$ metric projection onto the closed and convex $\mathbf{C_i \subseteq H}$.

Image recovery problem: find a $\mathbf{p \in C_0 := \cap_{1 \leq i \leq r} C_i}$.

Define $\mathbf{T_i := I + \lambda_i(P_i - I)}$ for $\mathbf{0 < \lambda_i \leq 2, \lambda_1 < 2}$ and $\mathbf{T := \sum_{i=1}^{r} \alpha_i T_i}$, where $\boldsymbol{\alpha_i \in (0,1),\ \sum \alpha_i = 1}$.

Crombez 1992: $\mathbf{Fix(T) = C_0}$ and $\mathbf{T}$ is asymptotically regular, if $\mathbf{C_0}$ is nonempty.

Let $C_{i,\varepsilon} := \cup_{x \in C_i} B_\varepsilon(x)$, $C_{0,\varepsilon} := \cap_{i=1}^{r} C_{i,\varepsilon}$.

Let $C_{i,\varepsilon} := \cup_{x \in C_i} B_\varepsilon(x),\ C_{0,\varepsilon} := \cap_{i=1}^r C_{i,\varepsilon}$.

Theorem (M.A.A. Khan/K., Nonlinear Analysis 2014)

Let $D > \|x_0 - p\|$ *for some* $p \in C_0$ *and* $N_1, N_2 \in \mathbb{N}$ *s.t.*

$$\frac{1}{N_1} \leq \min\{\alpha_i\lambda_i : 1 \leq i \leq r\},\ \frac{1}{N_2} \leq \min\{\alpha_1, 2 - \lambda_1\}.$$

Then for $x_n := T^{(n)}x_0,\ x_0 \in H$*:*

$$\forall \varepsilon \in (0,1)\, \forall n \geq \Psi(D, N_1, N_2, \varepsilon)\ (x_n \in C_{0,\varepsilon}),$$

Let $\mathbf{C_{i,\varepsilon} := \cup_{x\in C_i} B_\varepsilon(x)}, \ \mathbf{C_{0,\varepsilon} := \cap_{i=1}^r C_{i,\varepsilon}}$.

Theorem (M.A.A. Khan/K., Nonlinear Analysis 2014)

Let $\mathbf{D > \|x_0 - p\|}$ *for some* $\mathbf{p \in C_0}$ *and* $\mathbf{N_1, N_2 \in \mathbb{N}}$ *s.t.*

$$\frac{1}{N_1} \leq \min\{\alpha_i\lambda_i : 1 \leq i \leq r\}, \ \frac{1}{N_2} \leq \min\{\alpha_1, 2-\lambda_1\}.$$

Then for $\mathbf{x_n := T^{(n)}x_0, \ x_0 \in H}$:

$$\forall \varepsilon \in (0,1)\, \forall n \geq \Psi(D, N_1, N_2, \varepsilon)\ (x_n \in C_{0,\varepsilon}),$$

where

$$\Psi(D, N_1, N_2, \varepsilon) := \left\lceil \frac{1936 \cdot N_1^6 \cdot (D+1)^4 (4N_1+1)^2 \cdot (2N_2+1)^2}{\pi \cdot \varepsilon^4} \right\rceil.$$

Convex minimization problem for compositions of maps

Theorem (Ariza-Ruiz/López-Acedo/Nicolae, JOTA 2015)

Let $\mathbf{X}$ be a CAT(0)-space and $\mathbf{T_1, T_2 : X \to X}$ satisfy the condition

$$\mathbf{(P): 2d(T_ix, T_iy)^2 \leq d(x, T_iy)^2 + d(y, T_ix)^2 - d(x, T_ix)^2 - d(y, T_iy)^2.}$$

Convex minimization problem for compositions of maps

Theorem (Ariza-Ruiz/López-Acedo/Nicolae, JOTA 2015)

Let $\mathbf{X}$ be a CAT(0)-space and $\mathbf{T_1, T_2 : X \to X}$ satisfy the condition

$$\mathbf{(P): 2d(T_i x, T_i y)^2 \leq d(x, T_i y)^2 + d(y, T_i x)^2 - d(x, T_i x)^2 - d(y, T_i y)^2.}$$

Let $\mathbf{Fix(T_2 \circ T_1) \neq \emptyset}$. Consider sequences $\mathbf{(x_n), (y_n)}$ in $\mathbf{X}$ with

$$\mathbf{d(y_n, T_1 x_n) \leq \varepsilon_n} \text{ and } \mathbf{d(x_{n+1}, T_2 y_n) \leq \delta_n}, \text{ for all } \mathbf{n} \in \mathbb{N},$$

where $\sum_{n=0}^{\infty} \varepsilon_n < \infty$ and $\sum_{n=0}^{\infty} \delta_n < \infty$.

Convex minimization problem for compositions of maps

Theorem (Ariza-Ruiz/López-Acedo/Nicolae, JOTA 2015)

Let $\mathbf{X}$ be a CAT(0)-space and $\mathbf{T_1, T_2 : X \to X}$ satisfy the condition

$$\mathbf{(P): 2d(T_i x, T_i y)^2 \leq d(x, T_i y)^2 + d(y, T_i x)^2 - d(x, T_i x)^2 - d(y, T_i y)^2.}$$

Let $\mathbf{Fix(T_2 \circ T_1) \neq \emptyset}$. Consider sequences $\mathbf{(x_n), (y_n)}$ in $\mathbf{X}$ with

$$\mathbf{d(y_n, T_1 x_n) \leq \varepsilon_n} \text{ and } \mathbf{d(x_{n+1}, T_2 y_n) \leq \delta_n}, \text{ for all } \mathbf{n} \in \mathbb{N},$$

where $\sum_{n=0}^{\infty} \varepsilon_n < \infty$ and $\sum_{n=0}^{\infty} \delta_n < \infty$.

Then

$$\mathbf{\lim d(y_{n+1}, y_n) = \lim d(x_{n+1}, x_n) = 0.}$$

Convex minimization problem for compositions of maps

Theorem (Ariza-Ruiz/López-Acedo/Nicolae, JOTA 2015)

Let $\mathbf{X}$ be a CAT(0)-space and $\mathbf{T_1, T_2 : X \to X}$ satisfy the condition

$\mathbf{(P)\colon 2d(T_i x, T_i y)^2 \le d(x, T_i y)^2 + d(y, T_i x)^2 - d(x, T_i x)^2 - d(y, T_i y)^2.}$

Let $\mathbf{Fix(T_2 \circ T_1) \neq \emptyset}$. Consider sequences $\mathbf{(x_n), (y_n)}$ in $\mathbf{X}$ with

$$\mathbf{d(y_n, T_1 x_n) \le \varepsilon_n} \text{ and } \mathbf{d(x_{n+1}, T_2 y_n) \le \delta_n}, \text{ for all } \mathbf{n} \in \mathbb{N},$$

where $\sum_{n=0}^{\infty} \varepsilon_n < \infty$ and $\sum_{n=0}^{\infty} \delta_n < \infty$.

Then

$$\mathbf{\lim d(y_{n+1}, y_n) = \lim d(x_{n+1}, x_n) = 0.}$$

Proof makes **repeated use** of convergence of mon. sequences (**ACA**)!

Application of the theorem

Consider two convex and lower semi-continuous $f, g : X \to (-\infty, +\infty]$ and define (Bauschke, Combettes, Reich 2005)

$$\Phi(x, y) := f(x) + g(y) + \frac{1}{2\lambda} d(x, y)^2.$$

Then $T_1 = J^g_\lambda$, $T_2 = J^f_\lambda$ (resolvents of f, g) satisfy (P).

Computing sequences $(x_n), (y_n)$ as above (which only requires to know the resolvents up to some error) provides ε-solutions for the minimization problem

$$\operatorname*{argmin}_{(x,y)\in X\times X} \Phi(x, y).$$

Rate of convergence in the theorem

Theorem (K./López-Acedo/Nicolae, Optimization 2016)

Let α be a Cauchy-rate for $\sum_{n=0}^{\infty} \gamma_n$ with $\gamma_n := \varepsilon_n + \delta_n$ and $\sum \gamma_n \leq B \in \mathbb{N}$ and $d(x_0, u) \leq b$ for some $u \in Fix(T_2 \circ T_1)$.

Rate of convergence in the theorem

Theorem (K./López-Acedo/Nicolae, Optimization 2016)

Let α be a Cauchy-rate for $\sum_{n=0}^{\infty} \gamma_n$ with $\gamma_n := \varepsilon_n + \delta_n$ and $\sum \gamma_n \leq B \in \mathbb{N}$ and $d(x_0, u) \leq b$ for some $u \in Fix(T_2 \circ T_1)$.
Then

$$\forall n \geq \Phi(\varepsilon, b, B, \alpha) \ (d(y_n, y_{n+1}), d(x_n, x_{n+1}) \leq \varepsilon),$$

where

$$\Phi := \alpha(\varepsilon/6) + k \left\lceil \frac{24(1+2^k)(b+B)}{\varepsilon} \right\rceil + 1, \ k := \left\lceil \frac{24(b+B)}{\varepsilon} \right\rceil.$$

$(x_n), (y_n)$ uniformly quasi-Fejér monotone, hence **rates of metastability** for **totally bounded X**:

$(x_n), (y_n)$ uniformly quasi-Fejér monotone, hence **rates of metastability** for **totally bounded** X:

Theorem (K./López-Acedo/Nicolae, Optimization 2016)

Let X additionally be **totally bounded** with a modulus γ. Then

$$\forall k \in \mathbb{N} \forall g \in \mathbb{N}^{\mathbb{N}} \exists n \leq \Psi(k,g) \forall i,j \in [n, n+g(n)] \left(d(x_i, x_j) \leq \frac{1}{k+1} \right),$$

$(x_n), (y_n)$ uniformly quasi-Fejér monotone, hence **rates of metastability** for **totally bounded** X:

Theorem (K./López-Acedo/Nicolae, Optimization 2016)

Let X additionally be **totally bounded** with a modulus γ. Then

$$\forall k \in \mathbb{N} \forall g \in \mathbb{N}^{\mathbb{N}} \exists n \leq \Psi(k,g) \forall i,j \in [n, n+g(n)] \left(d(x_i, x_j) \leq \frac{1}{k+1} \right),$$

where

$$\Psi(k,g) := \Psi_0(P),\ P := \gamma(8k+7)+1,\ \xi(k) := \alpha(1/(k+1)),$$

$$\chi_g^M(n,k) := (\max_{i \leq n} g(i)) \cdot (k+1),$$

$\mathbf{(x_n), (y_n)}$ uniformly quasi-Fejér monotone, hence **rates of metastability** for **totally bounded X**:

Theorem (K./López-Acedo/Nicolae, Optimization 2016)

Let X additionally be **totally bounded** with a modulus γ. Then

$$\forall k \in \mathbb{N} \forall g \in \mathbb{N}^{\mathbb{N}} \exists n \leq \Psi(k,g) \forall i,j \in [n, n+g(n)] \left(d(x_i, x_j) \leq \frac{1}{k+1} \right),$$

where

$$\Psi(k,g) := \Psi_0(P),\ P := \gamma(8k+7)+1,\ \xi(k) := \alpha(1/(k+1)),$$

$$\chi_g^M(n,k) := (\max_{i \leq n} g(i)) \cdot (k+1),$$

and using $\widehat{\Phi}(k,N) := \max\{N, \Phi(1/2(i+1)) : i \leq k\}$)

$$\Psi_0(0) := 0,\ \Psi_0(n+1) := \widehat{\Phi}\left(\chi_g^M(\Psi_0(n), 8k+7), \xi(8k+7)\right).$$

$(x_n), (y_n)$ uniformly quasi-Fejér monotone, hence **rates of metastability** for **totally bounded X**:

Theorem (K./López-Acedo/Nicolae, Optimization 2016)

Let X additionally be **totally bounded** with a modulus γ. Then

$$\forall k \in \mathbb{N} \forall g \in \mathbb{N}^{\mathbb{N}} \exists n \leq \Psi(k,g) \forall i,j \in [n, n+g(n)] \left(d(x_i, x_j) \leq \frac{1}{k+1} \right),$$

where

$$\Psi(k,g) := \Psi_0(P),\ P := \gamma(8k+7)+1,\ \xi(k) := \alpha(1/(k+1)),$$

$$\chi_g^M(n,k) := (\max_{i \leq n} g(i)) \cdot (k+1),$$

and using $\widehat{\Phi}(k,N) := \max\{N, \Phi(1/2(i+1)) : i \leq k\})$

$$\Psi_0(0) := 0,\ \Psi_0(n+1) := \widehat{\Phi}\left(\chi_g^M(\Psi_0(n), 8k+7), \xi(8k+7)\right).$$

Similar results for the famous Rockafellar **Proximal Point Algorithm** (K./Leuştean/Nicolae 2015).

I. Yamada's Theorem on a Variational Inequality Problem

Problem: H real Hilbert space, $\Theta : H \to \mathbb{R}$, solve $\min \Theta$ over closed convex $C \subseteq X$.

Let gradient $F := \Theta'$ be κ-Lipschitzian and η-strongly monotone and $C = Fix(T)$ for some nonexpansive $T : H \to H$.

I. Yamada's Theorem on a Variational Inequality Problem

Problem: H real Hilbert space, $\Theta : H \to \mathbb{R}$, solve $\min \Theta$ over closed convex $C \subseteq X$.

Let gradient $F := \Theta'$ be κ-Lipschitzian and η-strongly monotone and $C = Fix(T)$ for some nonexpansive $T : H \to H$.

Equivalent formulation:

$$\text{VIP: Find } \mathbf{u^*} \in \mathbf{C} \text{ s.t. } \langle \mathbf{v} - \mathbf{u^*}, \mathbf{F(u^*)} \rangle \geq \mathbf{0} \text{ for all } \mathbf{v} \in \mathbf{C}.$$

I. Yamada's Theorem on a Variational Inequality Problem

Problem: H real Hilbert space, $\Theta : H \to \mathbb{R}$, solve $\min \Theta$ over closed convex $C \subseteq X$.

Let gradient $F := \Theta'$ be κ-Lipschitzian and η-strongly monotone and $C = Fix(T)$ for some nonexpansive $T : H \to H$.

Equivalent formulation:

$$\text{VIP: Find } \mathbf{u^*} \in \mathbf{C} \text{ s.t. } \langle \mathbf{v} - \mathbf{u^*}, \mathbf{F(u^*)} \rangle \geq \mathbf{0} \text{ for all } \mathbf{v} \in \mathbf{C}.$$

Theorem (I. Yamada 2001): Under suitable conditions on (λ_n) the scheme (with $\mu := \eta/\kappa^2$)

$$\mathbf{u_{n+1} := T(u_n) - \lambda_{n+1} \mu F T(u_n)}$$

converges strongly to a solution of VIP.

I. Yamada's Theorem on a Variational Inequality Problem

Problem: H real Hilbert space, $\Theta : H \to \mathbb{R}$, solve $\min \Theta$ over closed convex $C \subseteq X$.

Let gradient $F := \Theta'$ be κ-Lipschitzian and η-strongly monotone and $C = Fix(T)$ for some nonexpansive $T : H \to H$.

Equivalent formulation:

$$\text{VIP: Find } \mathbf{u^*} \in \mathbf{C} \text{ s.t. } \langle \mathbf{v} - \mathbf{u^*}, \mathbf{F(u^*)} \rangle \geq \mathbf{0} \text{ for all } \mathbf{v} \in \mathbf{C}.$$

Theorem (I. Yamada 2001): Under suitable conditions on (λ_n) the scheme (with $\mu := \eta/\kappa^2$)

$$\mathbf{u_{n+1} := T(u_n) - \lambda_{n+1}\mu F T(u_n)}$$

converges strongly to a solution of VIP.

Theorem (D. Körnlein 2016): Explicit and highly uniform effective rate of metastability. Simple rate of asymptotic regularity.

Literature

1) Ariza-Ruiz, D., Leuştean, L., López-Acedo, G., Firmly nonexpansive mappings in classes of geodesic spaces. Trans. Amer. Math. Soc. **366**, pp. 4299-4322 (2014).

2) Ariza-Ruiz, D., López-Acedo, G., Nicolae, A., The asymptotic behavior of the composition of firmly nonexpansive mappings. J. Optim. Theory Appl. **167**, pp. 409-429 (2015).

3) Briseid, E.M., A rate of convergence for asymptotic contractions. J. Math. Anal. Appl. **330**, pp. 364-376 (2007).

4) Briseid, E.M., Some results on Kirk's asymptotic contractions. Fixed Point Theory **8**, pp. 17-27 (2007).

5) Briseid, E.M., Fixed points of generalized contractive mappings. J. of Nonlinear and Convex Anal. **9**, pp. 181-204 (2008).

6) Briseid, E.M., Logical aspects of rates of convergence in metric spaces. J. Symb. Logic **74**, pp. 1401-1428 (2009).

7) Colao, V., Leuştean, L., Lopez, G., Martin-Marquez, V., Alternative iterative methods for nonexpansive mappings, rates of convergence and application. Journal of Convex Analysis **18**, pp. 465-487 (2011).

8) Gerhardy, P., A quantitative version of Kirk's fixed point theorem for asymptotic contractions. J. Math. Anal. Appl. **316**, pp. 339-345 (2006).

9) Ivan, D., Leuştean, L., A rate of asymptotic regularity for the Mann iteration of κ-strict pseudo-contractions, Numer. Funct. Anal. Optimiz. **36**, pp. 792-798 (2015).

10) Khan, M.A.A., Kohlenbach, U., Bounds on Kuhfittig's iteration schema in uniformly convex hyperbolic spaces. J. Math. Anal. Appl. **403**, pp. 633-642 (2013).

11) Khan, M.A.A., Kohlenbach, U., Quantitative image recovery theorems. Nonlinear Anal. **106**, pp. 138-150 (2014).

12) Kohlenbach, U., On the computational content of the Krasnoselski and Ishikawa fixed point theorems. In: Proceedings of the Fourth Workshop on Computability and Complexity in Analysis, J. Blanck, V. Brattka, P. Hertling (eds.), Springer LNCS **2064**, pp. 119-145 (2001).

13) Kohlenbach, U., A quantitative version of a theorem due to Borwein-Reich-Shafrir. Numer. Funct. Anal. and Optimiz. **22**, pp. 641-656 (2001).

14) Kohlenbach, U., Uniform asymptotic regularity for Mann iterates. J. Math. Anal. Appl. **279**, pp. 531-544 (2003).

15) Kohlenbach, U., Some computational aspects of metric fixed point theory. Nonlinear Analysis **61**, pp. 823-837 (2005).

16) Kohlenbach, U., On the asymptotic behavior of odd operators. J. Math. Anal. Appl. **382**, pp. 615-620 (2011).

17) Kohlenbach, U., On the quantitative asymptotic behavior of strongly nonexpansive mappings in Banach and geodesic spaces. To appear in: Israel Journal of Mathematics.

18) Kohlenbach, U., Koutsoukou-Argyraki, A., Rates of convergence and metastability for abstract Cauchy problems generated by accretive operators. J. Math. Anal. Appl. **423**, 1089-1112 (2015).

19) Kohlenbach, U., Koutsoukou-Argyraki, A., Effective asymptotic regularity for one-parameter nonexpansive semigroups. J. Math. Anal. Appl. **433**, pp. 1883-1903 (2016).

20) Kohlenbach, U., Lambov, B., Bounds on iterations of asymptotically quasi-nonexpansive mappings. In: G-Falset, J., L-Fuster, E., Sims, B. (eds.), Proc. International Conference on Fixed Point Theory, Valencia 2003, pp. 143-172, Yokohama Press, 2004.

21) Kohlenbach, U., Leuştean, L., Mann iterates of directionally nonexpansive mappings in hyperbolic spaces. Abstract and Applied Analysis, vol. 2003, no.8, pp. 449-477 (2003).

22) Kohlenbach, U., Leuştean, L., The approximate fixed point property in product spaces. Nonlinear Analysis **66**, pp. 806-818 (2007).

23) Kohlenbach, U., Leuştean, L., Asymptotically nonexpansive mappings in uniformly convex hyperbolic spaces. Journal of the European Mathematical Society **12**, pp. 71-92 (2010).

24) Kohlenbach, U., Leuştean, L., Nicolae, A., Quantitative results of Fejér monotone sequences. Preprint 2014, arXiv:1412.5563, submitted.

25) Kohlenbach, U., López-Acedo, G., Nicolae, A., Quantitative asymptotic regularity for the composition of two mappings. To appear in: Optimization.

26) Körnlein, D., Quantitative Analysis of Iterative Algorithms in Fixed Point Theory and Convex Optimization. PhD Thesis, TU Darmstadt 2016.

27) Körnlein, D., Quantitative results for Halpern iterations of nonexpansive mappings. J. Math. Anal. Appl. **428**, pp. 1161-1172 (2015).

28) Körnlein, D., Quantitative results for Bruck iterations of demicontinuous pseudocontractions. Preprint 2015, submitted.

29) Körnlein, D., Kohlenbach, U., Effective rates of convergence for Lipschitzian pseudocontractive mappings in general Banach spaces. Nonlinear Analysis **74**, pp. 5253-5267 (2011).

30) Körnlein, D., Kohlenbach, U., Rates of metastability for Bruck's iteration of pseudocontractive mappings in Hilbert space. Numer. Funct. Anal. Optimiz. **35**, pp. 20-31 (2014).

31) Leuştean, L., A quadratic rate of asymptotic regularity for CAT(0)-spaces. J. Math. Anal. Appl. **325**, pp. 386-399 (2007).

32) Leuştean, L., Nonexpansive iterations in uniformly convex W-hyperbolic spaces in A. Leizarowitz, B. S. Mordukhovich, I. Shafrir, A. Zaslavski (Editors): Nonlinear Analysis and Optimization I: Nonlinear Analysis, Contemporary Mathematics, Vol. 513 (2010), AMS, 193-209.

33) Leuştean, L., An application of proof mining to nonlinear iterations. Ann. Pure Appl. Logic **165**, pp. 1484-1500 (2014).

34) Leuştean, L., Nicolae, A., Effective results on compositions of nonexpansive mappings. J. Math. Anal. Appl. **410**, pp. 902-907 (2014).

35) Neumann, E., Computational problems in metric fixed point theory and their Weihrauch degrees. Log. Method. Comput. Sci. **11**, 44pp. (2015).

36) Nicolae, A., Asymptotic behavior of averaged and firmly nonexpansive mappings in geodesic spaces. Nonlinear Analysis **87**, pp. 102-115 (2013).

37) Sipos, A., A note on the Mann iteration for k-strict pseudocontractions in Banach spaces. Preprint 2016, arXiv:1605.02237.

38) Sipos, A., Effective results on a fixed point algorithm for families of nonlinear mappings. Preprint 2016, arXiv:1606.03895.

2016 survey:

www.mathematik.tu-darmstadt.de/~kohlenbach/progress.pdf

2008 book: