Recursion and complexity
(2nd lecture)

Isabel Oitavem

CMA and DM, FCT-UNL

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Recursion-theoretic approaches

- One works over $\mathbb{W} =< \epsilon, S_0, S_1 >$.
- Class of initial functions $\mathcal{I}$:
  - ϵ, S_0 and S_1 (the constructores of the algebra);
  - P (predecessor);
  - C (case distinction);
  - π_j^n (projections).

[$\mathcal{I}$; **Composition**, **Recursion schemes**]

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- $f = \mathbf{COMP}[g, \bar{h}]$ i.e. $f(\bar{x}) = g(\bar{h}(\bar{x}))$.

Bounded recursion-theoretic approach: **FPtime**

FPtime $= [\mathcal{I}; \textbf{COMP}, \textbf{BR}]$ (Cobham 1964)

BR (**Bounded recursion over** $\mathbb{W}$):

$$
\begin{aligned}
f(\epsilon, \bar{x}) &= g(\epsilon, \bar{x}) \\
f(y0, \bar{x}) &= h(y0, \bar{x}, f(y, \bar{x}))_{|t(y0,\bar{x})} \\
f(y1, \bar{x}) &= h(y1, \bar{x}, f(y, \bar{x}))_{|t(y1,\bar{x})}
\end{aligned}
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t is a bounding function, i.e. t is explicitly definable from ϵ, S_0, S_1, string concatenation and string product.

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BPR (**Bounded recursion over** $\mathbb{W}$):

$$f(\epsilon, \bar{x}) = g(\epsilon, \bar{x})$$
$$f(y+1, \bar{x}) = h(y, \bar{x}, f(y, \bar{x}))_{|t(y+1,\bar{x})}$$

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- String concatenation: $\oplus(y, x) = xy, \qquad |\oplus(x, x)| = 2|x|;$
- String product: $\otimes(y, x) = \underbrace{x \cdots x}_{|y|-\text{times}}, \qquad |\otimes(x, x)| = |x|^2.$

Example: For $p(X) = X^2 + X + 1$, consider the bounding function $t(x) = S_1(\oplus(x, \otimes(x, x)))$. One has $|t(x)| = p(|x|)$.

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For every polynomial p, there exists a bounding function t such that $|t(x)| = p(|x|)$. And, vice-versa. <<

Exercises

1. For $p(X) = X + 1$, describe two different bounding functions t_0 and t_1, such that $|t_0(x)| = |t_1(x)| = p(|x|)$.
2. Define $\oplus$ in the Cobham's algebra $[\mathcal{I}; \mathrm{COMP}, \mathrm{BR}]$.
3. Define $\otimes$ in the Cobham's algebra $[\mathcal{I}; \mathrm{COMP}, \mathrm{BR}]$.