Recursion and complexity
(3rd lecture)

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Implicit recursion-theoretic approach

Functions have two sorts of input positions, normal and safe: $f(\bar{x};\bar{y})$.

Input-sorted initial functions $\mathcal{SI}$:

- ϵ, $S_0(;x)$ and $S_1(;x)$ (the constructores of the algebra);
- $P(;x)$ (predecessor);
- $C(;x,y,z_0,z_1)$ (case distinction);
- $\pi_j^{m;n}$ (projections over both input sorts).

Input-sorted composition SC: $f(\bar{x};\bar{y}) = g(\bar{r}(\bar{x};)\,;\,\bar{s}(\bar{x};\bar{y}))$.

$[\,\mathcal{SI};$ **SC**, **Input-sorted Recursion** $]$

FPtime FPspace

Implicit recursion-theoretic approach: **FPtime**

$\textbf{FPtime} = [\,\mathcal{SI}; \textbf{SC}, \textbf{SR}\,]$ (Bellantoni-Cook 1992)

SR **(Input-sorted recursion over $\mathbb{W}$)**:

$$
\begin{aligned}
f(\epsilon, \bar{x}; \bar{y}) &= g(\epsilon, \bar{x}; \bar{y}) \\
f(z0, \bar{x}; \bar{y}) &= h(z0, \bar{x}; \bar{y}, f(z, \bar{x}; \bar{y})) \\
f(z1, \bar{x}; \bar{y}) &= h(z1, \bar{x}; \bar{y}, f(z, \bar{x}; \bar{y}))
\end{aligned}
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Example: $f(11)$ leads to $h(11, h(1, g(\epsilon)))$

h

|

h

|

g

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|
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SR reproduces the sequential structure of deterministic computations.

<<

Implicit recursion-theoretic approach

Input-sorted composition SC: $f(\bar{x};\bar{y}) = g(\bar{r}(\bar{x};)\,;\,\bar{s}(\bar{x};\bar{y}))$.

If $F(x;y)$ is in the class then

- $f(x,y;) = F(x;y)$ is in the class;
- $f(;x,y) = F(x;y)$ is NOT in the class.

<<

Implicit recursion-theoretic approach: **FPtime**

SR:

$$f(\epsilon, \bar{x}; \bar{y}) = g(\epsilon, \bar{x}; \bar{y})$$
$$f(z0, \bar{x}; \bar{y}) = h(z0, \bar{x}; \bar{y}, f(z, \bar{x}; \bar{y}))$$
$$f(z1, \bar{x}; \bar{y}) = h(z1, \bar{x}; \bar{y}, f(z, \bar{x}; \bar{y}))$$

- Concatenation:
$$\oplus(\epsilon; x) = x$$
$$\oplus(y0; x) = S_0(; \oplus(y; x))$$
$$\oplus(y1; x) = S_1(; \oplus(y; x))$$

$\oplus(y, x;)$ is in the class.
$\oplus(; y, x)$ is NOT in the class.

Implicit recursion-theoretic approach: **FPtime**

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- Concatenation: $$\oplus(\epsilon; x) = x$$ $$\oplus(y0; x) = S_0(; \oplus(y; x))$$ $$\oplus(y1; x) = S_1(; \oplus(y; x))$$

  $\oplus(y, x;)$ is in the class.
  $\oplus(; y, x)$ is NOT in the class.
- exp such that $|\exp(z)| = 2^{|z|}$
$$\exp(\epsilon) = 1$$
$$\exp(zi) = \oplus(\exp(z), \exp(z))$$

Implicit recursion-theoretic approach: **FPtime**

SR:

$$f(\epsilon, \bar{x}; \bar{y}) = g(\epsilon, \bar{x}; \bar{y})$$
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PROBLEM!