Mahler measures: from small heights to big conjectures

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Abstract

The aim of this mini-course, given in occasion of the spring school *Zeta functions, dynamics and analytic number theory*, is to give an introduction to Mahler measures and to their relations with special values of L-functions.

Plan of the course

The protagonist of these four lectures is the *Mahler measure* of Laurent polynomials $P \in \mathbb{C}[x_1^{\pm 1}, \dots, x_n^{\pm 1}] \setminus \{0\}$, defined as

$$m(P) := \int_{\mathbb{T}^n} \log|P| d\mu_n = \int_{[0,1]^n} \log|P(e^{2\pi i t_1}, \dots, e^{2\pi i t_n})| dt_1 \cdots dt_n, \tag{0.1}$$

where $\mathbb{T}^n := \{(x_1, \dots, x_n) \in (\mathbb{C}^\times)^n \colon |x_1| = \cdots = |x_n| = 1\}$ and $\mu_n := \frac{1}{(2\pi i)^n} \frac{dx_1}{x_1} \wedge \cdots \wedge \frac{dx_n}{x_n}$.

Despite its simplicity, or maybe because of this, the Mahler measure of polynomials, which was introduced in [49], is ubiquitous in different areas of mathematics, ranging from Diophantine geometry to dynamics, from combinatorics to transcendence theory.

During the course of these four lectures, we will see how the Mahler measure can:

- detect the distribution of the roots of a univariate polynomial;
- compute the entropy of the actions of $\mathbb{Z}^n$ on compact, abelian groups;
- compute the growth of homology in towers of hyperbolic three-dimensional manifolds, and the growth of the number of spanning trees in towers of finite graphs;
- be related to special values of L-functions attached to different geometric objects.

Suggested reading

As the reader will immediately notice, these notes owe a large debt to several references mentioned within the text. In particular, we invite the interested reader to plunge into the books of Everest and Ward [27], Brunault and Zudilin [13], and McKee and Smyth [50], which provide excellent introductory references to the world of Mahler measures. Moreover, we invite the readers interested in dynamical systems to consult Schmidt's excellent monograph [62], and the readers interested in connections between arithmetic geometry, knots and graphs to consult Morishita's [51] and Terras's [71] excellent monographs. Finally, we invite the readers who are interested in the connections between Mahler measures and special values of L-functions associated to arithmetic varieties to dive into André's [1] and Huber and Müller-Stach's [35] monographs.

URL: https://drive.google.com/file/d/1C9JrhEk1v6p06Rg8eXzi7tm9oQT6eHKI/view?usp=sharing

Warning

Despite the author's revision efforts, these notes may very well contain some remaining typographical or more substantial errors. We apologize for any such occurrence with all the readers, and we kindly invite them to signal any mistake to our attention.

1 Univariate Mahler measures

In this first lecture, we concentrate on univariate Laurent polynomials $P \in \mathbb{C}[x^{\pm 1}] \setminus \{0\}$. In this case, one can express $m(P)$ in terms of the roots of P, using a classical result of Jensen [36].

Lemma 1.1 (Jensen). *For every $\alpha \in \mathbb{C}$ we have that $m(x-\alpha) = \log\max(|\alpha|, 1)$.*

Proof. The result is trivially true if $\alpha = 0$. If this is not the case, then $\alpha = re^{2\pi i\theta}$ for some $r > 0$ and $\theta \in [0, 1]$, which implies that

$$m(x-\alpha) = \int_0^1 \log|e^{2\pi it} - re^{2\pi i\theta}|dt = \int_0^1 \log|e^{2\pi is} - r|ds,$$

where $s = (t-\theta) \bmod 1$. Then we have that

$$\begin{aligned} m(x-\alpha) &= \lim_{n\to+\infty} \frac{1}{n}\sum_{k=1}^{n} \log|e^{2\pi ik/n} - r| = \lim_{n\to+\infty} \frac{1}{n} \log\left|\prod_{k=1}^{n}(e^{2\pi ik/n} - r)\right| \\ &= \lim_{n\to+\infty} \frac{\log|1-r^n|}{n} = \log\max(r, 1) = \log\max(|\alpha|, 1), \end{aligned}$$

as we wanted to show. □

Corollary 1.2 (Mahler). *If $P = x^b a_0(x-\alpha_1)\cdots(x-\alpha_d) \in \mathbb{C}[x^{\pm 1}] \setminus \{0\}$ then*

$$m(P) = \log|a_0| + \sum_{j=1}^{d} \log\max(|\alpha_j|, 1),$$

which implies that $m(P) \geq \log|a_0|$. In particular, $m(P) \geq 0$ if $P \in \mathbb{Z}[x^{\pm 1}] \setminus \{0\}$.

Proof. Write $\log|P| = b\log|x| + \log|a_0| + \sum_{j=1}^{d} \log|x-\alpha_j|$, and use Lemma 1.1. □

Since $m(P) \geq 0$ for every $P \in \mathbb{Z}[x] \setminus \{0\}$, it is natural to ask for which polynomials $P \in \mathbb{Z}[x] \setminus \{0\}$ the Mahler measure vanishes. This is answered by a classical result of Kronecker [39].

Proposition 1.3 (Kronecker). *Given an irreducible polynomial $P \in \mathbb{Z}[x] \setminus \{0\}$ we have that $m(P) = 0$ if and only if $P = x$ or P is a cyclotomic polynomial.*

Proof. It is clear from Corollary 1.2 that the Mahler measure of $P = x$ and of all cyclotomic polynomials vanishes, since all their roots lie on the unit circle. On the other hand, suppose that $P \in \mathbb{Z}[x] \setminus \{0\}$ is irreducible and that $m(P) = 0$. Write $P = a_0(x-\alpha_1)\cdots(x-\alpha_d)$ for $\alpha_1, \ldots, \alpha_d \in \overline{\mathbb{Q}}$ and $a_0 \in \mathbb{Z}$. Then $a_0 \in \{\pm 1\}$, because $\log|a_0| \leq m(P) = 0$, and therefore $\alpha_1, \ldots, \alpha_d$ are all algebraic integers lying in the closed unit disk. Now, for every $N \in \mathbb{N}$ we let $P_N(x) := a_0^N \prod_{j=1}^{d}(x-\alpha_j^N)$. Since for every $N \in \mathbb{N}$ we have that $|\alpha_j^N| = |\alpha_j|^N \leq 1$, the coefficients of P_N are bounded in absolute value, thanks to Viéte's formulas, and are all rational integers, thanks to Galois theory. Therefore $P_N = P_M$ for some $N, M \in \mathbb{N}$ such that $M > N$, which means that the multisets $\{\alpha_1^N, \dots, \alpha_d^N\}$ and $\{\alpha_1^M, \dots, \alpha_d^M\}$ coincide. By rearranging $\alpha_2, \dots, \alpha_d$, we can assume that $\alpha_1^N = \alpha_2^M$. Continuing like this, we see that there exists an integer $l \in \{1, \dots, d\}$ such that $\alpha_j^N = \alpha_{j+1}^M$ for every integer $j \in \{1, \dots, l-1\}$, while $\alpha_l^N = \alpha_1^M$. This implies that $\alpha_1^{N^l} = \alpha_1^{M^l}$, and therefore that either $\alpha_1 = 0$ or α_1 is a root of unity of order dividing $M^l - N^l$. Since P is irreducible, we can conclude that either $P = x$ or P is a cyclotomic polynomial, as we wanted to show. □

The polynomials P_N appearing in the proof of Proposition 1.3 have also been used by Pierce and Lehmer to define a sequence

$$\Delta_N(P) := (-1)^d P_N(1) = a_0^n \prod_{j=1}^{d} (\alpha_j^N - 1) \tag{1.4}$$

associated to any given univariate polynomial $P = a_0(x - \alpha_1) \cdots (x - \alpha_d) \in \mathbb{C}[x]$. When $P \in \mathbb{Z}[x]$ we see that $\Delta_N(P) \in \mathbb{Z}$ for every $N \in \mathbb{N}$.

Example 1.5. $\Delta_N(x-2) = 2^N - 1$ is the sequence of Mersenne numbers. In particular, $\Delta_N(x-2)$ is prime if and only if $N = 2$ or N is an odd prime and $\Delta_N(x-2)$ divides s_{N-2}, where $\{s_k\}_{k=0}^{+\infty} \subseteq \mathbb{N}$ is the sequence defined by the recurrence $s_k = s_{k-1}^2 - 2$, with initial value $s_0 = 4$.

The primality test outlined in Example 1.5, discovered by Lucas and Lehmer [45], can, to some extent, be generalized to the sequences $\Delta_N(P)$ associated to any polynomial $P \in \mathbb{Z}[x]$, as shown by Lehmer [46]. However, in order to apply these tests in practice, the sequence $|\Delta_N(P)|$ should not grow too fast. This led Lehmer to relate the growth of these sequences to the Mahler measure of P, as shown by the following theorem.

Theorem 1.6 (Lehmer). *For every $P \in \overline{\mathbb{Q}}[x] \setminus \{0\}$ that does not vanish on any root of unity, we have that*

$$\lim_{n \to +\infty} \frac{1}{N} \log|\Delta_N(P)| = m(P), \tag{1.7}$$

and moreover that $\lim_{N \to +\infty} |\Delta_{N+1}(P)/\Delta_N(P)| = \exp(m(P))$ if P does not vanish on the unit circle $\mathbb{T}$.

Proof. We can assume without loss of generality that $P = x - \alpha$ for some $\alpha \in \overline{\mathbb{Q}}$. If $|\alpha| \neq 1$ then

$$\lim_{N \to +\infty} |\alpha^N - 1|^{1/N} = \max(|\alpha|, 1) = \lim_{N \to +\infty} \left| \frac{\alpha^{N+1} - 1}{\alpha^N - 1} \right|,$$

which allows us to show the second part of our statement. For the first one, we have to see what happens if $|\alpha| = 1$, but α is not a root of unity. In this case, a result of Gelfond, which follows as well from Baker's celebrated theorem on linear forms in logarithms (see [27, Lemma 1.11]), assures us that there exist two constants $A, B \in \mathbb{R}_{>0}$, which depend solely on α, such that

$$A/n^B < |\alpha^n - 1| \leq 2$$

for every $n \geq 1$. Therefore, we have that

$$0 = \lim_{n \to +\infty} \frac{\log(A) - B\log(n)}{n} \leq \frac{\log|\alpha^n - 1|}{n} \leq \lim_{n \to +\infty} \frac{2}{n} = 0,$$

which allows us to conclude that (1.7) holds also when $|\alpha| = 1$. □

Theorem 1.6 led Lehmer to search for polynomials with integer coefficients and small, non-zero Mahler measure (see [46, Section 13]). Proceeding by increasing degree, he got to ask the following question.

Question 1.8 (Lehmer). *Does there exist a polynomial $L \in \mathbb{Z}[x] \setminus \{0\}$ such that $m(L) > 0$ and $m(L) \leq m(P)$ for every $P \in \mathbb{Z}[x] \setminus \{0\}$ such that $m(P) \neq 0$? If so, can one take $L = x^{10} + x^9 - x^7 - x^6 - x^5 - x^4 - x^3 + x + 1$?*

Surprisingly enough, not only is Question 1.8 open to this day, but also the specific polynomial

$$x^{10} + x^9 - x^7 - x^6 - x^5 - x^4 - x^3 + x + 1$$

still detains the record of the smallest known, non-zero Mahler measure. Nevertheless, there are several known partial results towards Lehmer's problem. The best known general lower bound for the Mahler measure of a polynomial with integer coefficients, due to Dobrowolski [24] and Voutier [74], fails short of proving Lehmer's conjecture only by a logarithmic factor in the degree of the polynomial.

Theorem 1.9 (Dobrowolski,Voutier). *Let $P \in \mathbb{Z}[x] \setminus \{0\}$ be an irreducible polynomial of degree $d \geq 2$. Then either*

$$m(P) \geq \frac{1}{4} \left(\frac{\log \log(d)}{\log(d)} \right)^3, \tag{1.10}$$

or $m(P) = 0$.

Proof. We will just give a sketch of the main ideas behind the proof of Dobrowolski [24], later simplified by Cantor and Sraus [14], and used by Voutier [74]. Let $\alpha_1, \ldots, \alpha_d$ be the roots of P. One can clearly assume that P is monic, and thus that $\alpha_1, \ldots, \alpha_d$ are algebraic integers. Moreover, one can also suppose without loss of generality that $\alpha_1^a \neq \alpha_j^a$ for every $j \geq 2$ and $a \in \mathbb{Z}_{\geq 1}$, because otherwise there exists a monic polynomial $Q \in \mathbb{Z}[x]$ such that $\deg(Q) < \deg(P)$ and $m(Q) = m(P)$. In fact, one can take Q to be the minimal polynomial of $\alpha_1 \cdots \alpha_k$, where $\{\alpha_1, \ldots, \alpha_k\}$ is an equivalence class for the equivalence relation that identifies α_i and α_j if and only if $\alpha_i^a = \alpha_j^a$.

Now, fix two integers $k, s \geq 1$ and let $p_1, \ldots, p_s$ be the first s prime numbers. From the previous observation, the sequence $(\alpha_1, \ldots, \alpha_d, \alpha_1^{p_1}, \ldots, \alpha_d^{p_1}, \ldots, \alpha_1^{p_s}, \ldots, \alpha_d^{p_s})$ consists of mutually distinct algebraic integers. This observation implies that, if $n = d(k+s)$, the $n \times n$ determinant

$$D := \det \left(\left(\binom{a-1}{b-1} \alpha_1^{a-b} \right)_{\substack{a=1,\ldots,n \\ b=1,\ldots,k}}, \ldots, \left(\binom{a-1}{b-1} \alpha_d^{a-b} \right)_{\substack{a=1,\ldots,n \\ b=1,\ldots,k}}, \left(\alpha_b^{(a-1)p_1} \right)_{\substack{a=1,\ldots,n \\ b=1,\ldots,d}}, \ldots, \left(\alpha_b^{(a-1)p_s} \right)_{\substack{a=1,\ldots,n \\ b=1,\ldots,d}} \right)$$

is a non-zero rational integer, which has the following explicit expression

$$\begin{aligned} D^2 = &\left(\prod_{1 \leq i < j \leq d} (\alpha_i - \alpha_j)^2 \right)^{k^2} \cdot \left(\prod_{r=1}^{s} \prod_{1 \leq i < j \leq d} (\alpha_i^{p_r} - \alpha_j) \right)^{2k} \cdot \left(\prod_{r=1}^{s} \prod_{1 \leq i < j \leq d} (\alpha_i^{p_r} - \alpha_j^{p_r})^2 \right) \\ &\cdot \left(\prod_{1 \leq r_1 < r_2 \leq s} \prod_{1 \leq i < j \leq d} (\alpha_i^{p_{r_1}} - \alpha_j^{p_{r_2}})^2 \right), \end{aligned} \tag{1.11}$$

as proven in [38, Theorem 20]. Looking at the second factor appearing in (1.11), we see that D is divisible by the rational integer $\prod_{r=1}^{s} \prod_{j=1}^{d} P(\alpha_j^{p_r})^k$, which is in turn a multiple of $\prod_{r=1}^{s} p_r^{dk}$, thanks to Fermat's little theorem. Moreover, looking at the first and third factors appearing in (1.11), Voutier shows that

$$D^2 \geq |\Delta_{K_P}|^{k^2+s} \cdot \prod_{r=1}^{s} p_r^{2dk}, \tag{1.12}$$

where Δ_{K_P} denotes the absolute discriminant of the number field $K_P := \mathbb{Q}[x]/(P)$. On the other hand, one can bound D^2 from above as follows

$$D^2 \leq n^{d(k^2+s)} \cdot \left(\prod_{i=0}^{k-1} (2i+1)!(i!)^2 \right)^{-d} \cdot \exp(m(P))^{(2n(k+p_1+\cdots+p_s))}, \tag{1.13}$$

using Hadamard's inequality [31, Theorem 30]. Combining (1.12) and (1.13) gives a lower bound for $m(P)$ in terms of the discriminant of K_P, the first s prime numbers, and the parameter k.

In order to make this lower bound for $m(P)$ completely explicit, Voutier uses the lower bound

$$|\Delta_{K_P}| \geq (\gamma + \log(4\pi)) \cdot d - 8.6 \cdot \sqrt[3]{d} \geq 3.108 \cdot d - 8.6 \cdot \sqrt[3]{d},$$

which is due to Odlyzko and Poitou [54, Equation 22], and the bounds

$$\begin{aligned} p_1 + \cdots + p_s &\leq 0.564 \cdot s^2 \cdot \log(s) && \text{if } s \geq 9 \\ \log(p_1) + \cdots + \log(p_s) &\geq s \log(s) && \text{if } s \geq 13 \end{aligned}$$

which follow from works of Robin [56] and Rosser [58]. More precisely, thanks to these results, Voutier observes that:

- if $22 \leq d \leq 10000$, one can take $k = 7$ and $s = 17$ to show that (1.10) holds;
- if $d > 10000$, one can take $k := k_1 \log(d)/\log\log(d)$ and $s := s_1(\log(d)/\log\log(d))^2$ for any two real numbers $k_1 \in [1.26, 1.51]$ and $s_1 \in [k_1 - 0.06, k_1]$ such that the resulting k and s are rational integers. Note that there exist two such numbers k_1 and s_1, because $d > 10000$.

Since the minimal values of the non-zero Mahler measures of integral polynomials $P \in \mathbb{Z}[x] \setminus \{0\}$ having degree $2 \leq d \leq 21$ are explicitly known thanks to work of Boyd [7], this allows Voutier to conclude. □

There are also several special classes of polynomials $P \in \mathbb{Z}[x]$ for which Question 1.8 is known to admit a positive answer. For example, Schinzel [60] has shown that the Mahler measure of polynomials whose roots are all real is bounded away from zero, and actually increases linearly with the degree.

Theorem 1.14 (Schinzel). *Let $P \in \mathbb{Z}[x] \setminus \{0\}$ be an irreducible, monic polynomial of degree $d \geq 2$ whose complex roots are all real. Then $m(P) \geq \frac{d}{2} \log\left(\frac{1+\sqrt{5}}{2}\right)$, with equality if and only if $P = x^2 - x - 1$.*

Proof. Since $P \in \mathbb{Z}[x]$ is irreducible, $|P(0)| \geq 1$, and $|P(1)P(-1)| \geq 1$. Therefore, setting

$$f(t) := |t|^{1/2} \cdot |t - t^{-1}|^{1/(2\sqrt{5})}$$

we see that $\prod_{j=1}^d f(\alpha_j) = |P(0)|^{1/2 - 1/(2\sqrt{5})} \cdot |P(1)P(-1)|^{1/(2\sqrt{5})} \geq 1$, where $\alpha_1, \ldots, \alpha_d$ are the complex roots of P. Since $\alpha_1, \ldots, \alpha_d$ are all real, to conclude it suffices to observe that

$$\log(f(t)) = \frac{1}{2} \log|t| + \frac{1}{2\sqrt{5}} \log\left| t - \frac{1}{t} \right| \leq \log\max(1, |t|) - \frac{1}{2} \log\left(\frac{\sqrt{5}+1}{2} \right) \tag{1.15}$$

for every $t \in \mathbb{R}^\times$. To prove this, we remark that the maximum of the function

$$g(t) := \log(f(t)) - \log\max(1, |t|)$$

is attained in the open interval $(0, 1)$, because $g(1/t) = g(t) = g(-t)$. Computing the derivative of $g(t)$ we see that this maximum is attained at the point $t_0 = \frac{\sqrt{5}-1}{\sqrt{5}+1}$, and is equal to $-\frac{1}{2} \log\left(\frac{\sqrt{5}+1}{2}\right)$, as desired. □

Moreover, Smyth [68] has shown that the Mahler measures of polynomials which are not self-reciprocal is bounded from below by the Mahler measure of the cubic polynomial $x^3 - x - 1$.

Theorem 1.16 (Smyth). *Let $P \in \mathbb{Z}[x] \setminus \{0\}$ be an irreducible polynomial of degree n, and suppose that $P \neq x^n P^*$, where $P^*(x) := P(1/x)$ is the reciprocal of P. Then $m(P) \geq m(x^3 - x - 1)$.*

Proof. We can clearly assume without loss of generality that $P(0) = \pm 1$ and that $m(P) \leq \log(4) - \log(3)$ because $\log(4) - \log(3) > m(x^3 - x - 1)$. Moreover, we note that P does not have any root on the unit circle, because otherwise P would be reciprocal.

Then, the basic idea behind Smyth's proof, which dates back to works of Salem and Siegel, is to look at the rational function

$$\psi(x) := \frac{P(0)P(x)}{x^n P^*(x)} = 1 + a_k x^k + a_l x^l + \dots,$$

where $a_k x^k$ and $a_l x^l$ are the first non-zero monomials appearing in the Taylor expansion of this rational function at $x = 0$. Now, let $\alpha_1, \dots, \alpha_n$ be the roots of P, and consider the rational functions

$$f(x) := \varepsilon_f \cdot \prod_{|\alpha_j|<1} \left(\frac{x - \alpha_j}{1 - \overline{\alpha}_j x} \right) = c + c_1 x + c_2 x^2 + \dots$$
$$g(x) := \varepsilon_g \cdot \prod_{|\alpha_j|>1} \left(\frac{1 - \overline{\alpha}_j x}{x - \alpha_j} \right) = d + d_1 x + d_2 x^2 + \dots$$

where $\varepsilon_f, \varepsilon_g \in \{\pm 1\}$ are signs chosen in such a way that $c > 0$ and $d > 0$. Then, it is easy to see that

$$\psi(x) = 1 + a_k x^k + a_l x^l + \dots = \frac{f(x)}{g(x)} = \frac{c + c_1 x + c_2 x^2 + \dots}{d + d_1 x + d_2 x^2 + \dots}$$

and that $c = d = \exp(m(P))^{-1}$. Moreover, since a_k is the first non-vanishing Taylor coefficient of $\psi(x)$, we have that $c_j = d_j$ for every $j \in \{1, \dots, k-1\}$, and that $c_k - d_k = a_k c$, which gives $c \leq 2\max(|c_k|, |d_k|)$ because $a_k \in \mathbb{Z}$, and therefore $|a_k| \geq 1$.

Now, Smyth observes that $\max(|c_k|, |d_k|) \leq 1 - c^2$. To prove this, let $\beta := c^{-1} \cdot \operatorname{sgn}(c_k)$. Applying Parseval's identity to the function $f(x)(\beta + x^k)$, we see that

$$\begin{aligned} c^2\beta^2 + \left(c^2 + \sum_{j=1}^{k-1} c_j^2 \right) + (c + \beta c_k)^2 + \sum_{i=1}^{+\infty} (c_i + \beta c_{k+i})^2 &= \frac{1}{2\pi} \int_0^{2\pi} |f(e^{it})|^2 \cdot |\beta + e^{itk}|^2 dt \\ &\leq \frac{1}{2\pi} \int_0^{2\pi} |\beta + e^{itk}|^2 dt = \beta^2 + 1, \end{aligned}$$

because $f(x)$ has absolute value one on the unit circle. Therefore, we have that

$$1 + \left(c + \frac{|c_k|}{c} \right)^2 = c^2\beta^2 + (c + \beta c_k)^2 \leq \beta^2 + 1 = 1 + \frac{1}{c^2},$$

which implies that $|c_k| \leq 1 - c^2$. Analogously, one proves that $|d_k| \leq 1 - d^2 = 1 - c^2$. Therefore, we see that $c \leq 2\max(|c_k|, |d_k|) \leq 2(1 - c^2)$, which already implies that $\exp(m(P)) = c^{-1} \geq \frac{1+\sqrt{17}}{4}$.

To conclude the proof, Smyth applies again Parseval's identity to obtain various bounds on the coefficients of $f(x)$ and $g(x)$, which finally yield upper bounds for c and lower bounds for $\exp(m(P))$. □

More recently, Borwein, Dobrowolski and Mossinghoff [5] have shown that Lehmer's problem can be solved for polynomials with odd coefficients.

Theorem 1.17 (Borwein, Dobrowolski & Mossinghoff)**.** *Let $P \in \mathbb{Z}[x]$ be a polynomial of degree d with no cyclotomic factors, whose non-zero coefficients are all odd. Then $m(P) \geq \frac{\log(5)}{4}\left(1 - \frac{1}{d+1}\right) \geq \frac{\log(5)}{8}$.*

Proof. We can assume without loss of generality that P is monic, because $4\log(2) \geq \log(5)$. Moreover, for every polynomial $Q = b_0 + b_1 x + \cdots + b_n x^n \in \mathbb{Z}[x]$, and every root α of $P(x)$ we have that

$$|Q(\alpha^{d+1})| \leq L(Q)\max(1, |\alpha|^{n(d+1)}),$$

where $L(Q) := |b_0| + \cdots + |b_n|$ denotes the *length* of Q. Therefore, if $\alpha_1, \dots, \alpha_d$ denote the roots of P, we have that

$$|\mathrm{Res}(P, Q(x^{d+1}))| = \prod_{j=1}^{d} |Q(\alpha_j)| \leq L(Q)^d \exp(m(P))^{(d+1)n}, \tag{1.18}$$

where $\mathrm{Res}(F, G)$ denotes the *resultant* of two polynomials F, G.

Now, observe that for every $k \geq 0$, the polynomial

$$P_k := (x^{2^k(d+1)} + 1) + \frac{(x+1)(x^{2^k(d+1)} - 1)}{x^{d+1} - 1} P(x) \in \mathbb{Z}[x]$$

has only *even* coefficients, because P has only *odd* non-zero coefficients. Therefore, we have that

$$|\mathrm{Res}(P, x^{2^k(d+1)} + 1)| = |\mathrm{Res}(P, P_k)| \geq 2^d,$$

because P does not have any cyclotomic factors, and thus is coprime with P_k. This implies that

$$|\mathrm{Res}(P, Q(x^{d+1}))| \geq 2^{\nu(Q)d} \tag{1.19}$$

for every polynomial $Q \in \mathbb{Z}[x]$ such that $Q(x^{d+1})$ is coprime with P, where $\nu(Q)$ denotes the sum of the multiplicities of the cyclotomic polynomials Φ_{2^k} as factors of Q. Combining (1.18) with (1.19) we get

$$m(P) \geq \frac{\nu(Q)\log(2) - \log(L(Q))}{\deg(Q)}\left(1 - \frac{1}{d+1}\right)$$

for every such polynomial Q. Replacing Q by Q^k for some $k \geq 1$, we see that

$$m(P) \geq \frac{\nu(Q)\log(2) - \log(L(Q^k)^{1/k})}{\deg(Q)}\left(1 - \frac{1}{d+1}\right),$$

because $\nu(Q^k) = k\nu(Q)$ and $\deg(Q^k) = k\deg(Q)$. Taking the limit as $k \to +\infty$, we see that

$$m(P) \geq \frac{\nu(Q)\log(2) - \log(\|Q\|_\infty)}{\deg(Q)}\left(1 - \frac{1}{d+1}\right),$$

where $\|Q\|_\infty$ denotes the maximum of Q on the unit circle, which coincides with the limit of the quantities $L(Q^k)^{1/k}$ as $k \to +\infty$, because $\|Q^k\|_\infty \leq L(Q^k) \leq \sqrt{1 + k\deg(Q)}\|Q^k\|_\infty$ and $\|Q^k\|_\infty = \|Q\|_\infty^k$. To conclude, one can take $Q(x) = (x^2+1)(x^2-1)^4$, for which $\deg(Q) = 10$ and $\nu(Q) = 9$, while $\|Q\|_\infty = 2^9 \cdot 5^{-3/2}$. □

Finally, a very recent result of Dimitrov [22] shows that a consequence of Lehmer's question, proposed by Schinzel and Zassenhaus [61], holds true.

Theorem 1.20 (Dimitrov)**.** *Let α be a non-zero algebraic integer of degree d, which is not a root of unity. Then there exists a conjugate α' of α such that $|\alpha'| \geq 2^{1/(4d)}$.*

Proof. Let $P = (x - \alpha_1) \cdots (x - \alpha_d)$ be the minimal polynomial of α, and let $P_2(x) := \prod_{j=1}^{d}(x - \alpha_j^2)$ and $P_4(x) := \prod_{j=1}^{d}(x - \alpha_j^4)$. We proceed by induction on d, seeing that the base of the induction is trivially true. Now, if P_2 is reducible, the degree of α^2 is strictly smaller than the degree of α, and we can conclude by applying the induction hypothesis to α^2.

Suppose now that $P_2(x)$ is irreducible. Then, Dimitrov looks at the following power series

$$F(x) := \sqrt{P_2^*(x^{-1}) \cdot P_4^*(x^{-1})},$$

which lies in $\mathbb{Z}[\![x^{-1}]\!]$ because $P_2 \equiv P_4 (\mathrm{mod}\ 4)$, as follows from a generalization of Fermat's little theorem. This power series has no poles except at $x = 0$, and its zeros are precisely at the numbers $\alpha_1^2, \dots, \alpha_d^2$ and $\alpha_1^4, \dots, \alpha_d^4$. Therefore, if one lets $\mathcal{K}$ be the "hedgehog" consisting of the union of all the segments connecting the origin with $\alpha_1^2, \dots, \alpha_d^2$ and $\alpha_1^4, \dots, \alpha_d^4$, one sees that $F(z)$ is regular on the complement of $\mathcal{K}$ in the Riemann sphere $\mathbb{C} \cup \{\infty\}$. Thanks to a classical result of Dubinin [26, Theorem 4.17], the *transfinite diameter* of $\mathcal{K}$ is at most $2^{-1/d} \cdot \max_j |\alpha_j|^4$. Therefore, if we assume by contradiction that $\max_j |\alpha_j| < 2^{1/(4d)}$ we see that the transfinite diameter of $\mathcal{K}$ is strictly smaller than one, which implies, thanks to another classical result on the rationality of power series [50, Theorem 4.6], originally due to Pólya and Carlson, that $F(x)$ is a rational function. Since the only pole of $F(x)$ is at the origin, we see that $F(x)$ is in fact a polynomial in x^{-1}. Using the fact that P_2 is supposed to be irreducible, and that $P_2 \cdot P_4$ is a perfect square because $x^d \cdot F(x)$ is a polynomial, we see that $\alpha^2 = \alpha_j^4$ for some $j \in \{1, \dots, d\}$, which implies that P is cyclotomic, thanks to an argument similar to the one appearing in the proof of Proposition 1.3. □

In particular, Theorem 1.20 shows that for every $P = a_0(x - \alpha_1) \cdots (x - \alpha_d) \in \mathbb{Z}[x]$ we have that either

$$\log|a_0| + d \cdot \max_{j=1,\dots,d} (\log|\alpha_j|) \geq \frac{\log(2)}{4} \tag{1.21}$$

or $m(P) = 0$. In other words, if we replace $m(P)$ with the quantity appearing on the left hand side of (1.21), which is clearly bigger, the first part of Lehmer's Question 1.8 has a positive answer.

2 Entropies of dynamical systems and limits of Mahler measures

In this lecture, we will relate the Mahler measure of a multivariate polynomial to the entropy of a certain dynamical system. Before doing so, let us see why the Mahler measure always converges, following [13, Proposition 3.1].

Proposition 2.1. *For every $P \in \mathbb{C}[x_1^{\pm 1}, \dots, x_n^{\pm 1}] \setminus \{0\}$, the Mahler measure $m(P)$ introduced in (0.1) is a well-defined real number.*

Proof. Write $P = \sum_{\mathbf{v} \in \mathbb{Z}^n} c_{\mathbf{v}}(P) \cdot \underline{x}_n^{\mathbf{v}}$, where $\mathbf{v} = (v_1, \dots, v_n)$ and $\underline{x}_n^{\mathbf{v}} = x_1^{v_1} \cdots x_n^{v_n}$. Moreover, let $N_P \subseteq \mathbb{R}^n$ denote the *Newton polytope* of P, which is the convex hull of all those points $\mathbf{v} \in \mathbb{Z}^n$ such that $c_{\mathbf{v}}(P) \neq 0$. Then, choose a vertex $\mathbf{v}_0 \in N_P$, a rational hyperplane $H \subseteq \mathbb{R}^n$ which meets N_P only at $\mathbf{v}_0$, and a $\mathbb{Z}$-basis $(\mathbf{h}_1, \dots, \mathbf{h}_{n-1})$ of $H \cap \mathbb{Z}^n$. Such a $\mathbb{Z}$-basis can be completed to a $\mathbb{Z}$-basis of $\mathbb{Z}^n$ by adding only one vector $\mathbf{h}_n$, because $\mathbb{Z}^n / (H \cap \mathbb{Z}^n)$ is torsion-free and has rank one. Then, if we choose the sign of $\mathbf{h}_n$ to point outside

N_P, we see that the affine map $g\colon \mathbb{Z}^n \to \mathbb{Z}^n$ defined by $g(\mathbf{v}) := \mathbf{v}_0 + [\mathbf{h}_1 \mid \cdots \mid \mathbf{h}_n]^{-1} \cdot (\mathbf{v} - \mathbf{v}_0)$ defines a new Laurent polynomial $Q := \sum_{\mathbf{v}\in\mathbb{Z}^n} c_{\mathbf{v}}(P) \cdot \underline{x}_n^{g(\mathbf{v})}$ such that $m(P) = m(Q)$, by the change of variables formula, and such that the leading coefficient of Q with respect to the variable x_n is a monomial $Q_0 := c \cdot \underline{x}_n^{\mathbf{v}_0}$, for some $c \in \mathbb{C}^\times$. Therefore, we can write Q as a polynomial in x_n with coefficients in $\mathbb{C}[x_1^{\pm 1}, \dots, x_{n-1}^{\pm 1}]$, and we can factor it as $Q(\underline{x}_n) = Q_0 \cdot \prod_{j=1}^d (x_n - \alpha_j(x_1, \dots, x_{n-1}))$ for some algebraic functions $\alpha_j(x_1, \dots, x_{n-1})$ which are continuous because they depend continuously on the coefficients of Q/Q_0, which are Laurent polynomials because Q_0 is a monomial. To conclude, we apply Jensen's formula Lemma 1.1 to see that

$$m(P) = m(Q) = \log|c| + \sum_{j=1}^d \int_{\mathbb{T}^{n-1}} \log\max(|\alpha_j(x_1, \dots, x_{n-1})|, 1)\, d\mu_{n-1},$$

which implies that $m(P)$ is well defined, because the functions $\alpha_1, \dots, \alpha_d$ are continuous. □

Remark 2.2. The proof of Proposition 2.1 immediately implies that the Mahler measure depends continuously on the coefficients of P if its Newton polytope is fixed. In fact, the same holds true when the total degree of P is bounded, as shown by Boyd [9].

Let us now show that the Mahler measure of a Laurent polynomial $P \in \mathbb{Z}[x_1^{\pm 1}, \dots, x_n^{\pm 1}]$ with integer coefficients is the entropy of a certain dynamical system. More precisely, if $P = \sum_{\mathbf{v}\in\mathbb{Z}^n} c_{\mathbf{v}}(P) \underline{x}_n^{\mathbf{v}}$, we can consider the compact group

$$X_P := \{(y_{\mathbf{v}})_{\mathbf{v}\in\mathbb{Z}^n} \in \mathbb{T}^{\mathbb{Z}^n} \colon \prod_{\mathbf{v}\in\mathbb{Z}^n} y_{\mathbf{v}+\mathbf{w}}^{c_{\mathbf{v}}(P)} = 1,\ \forall \mathbf{w} \in \mathbb{Z}^n\},$$

endowed with the $\mathbb{Z}^n$-action α given by the shift, *i.e.* $\alpha_{\mathbf{w}}((y_{\mathbf{v}})_{\mathbf{v}\in\mathbb{Z}^n}) := (y_{\mathbf{w}+\mathbf{v}})$. Our aim for this lecture is to explain some ideas revolving around the proof of the following theorem, which is due to Lind, Schmidt and Ward [48].

Theorem 2.3 (Lind,Schmidt,Ward)**.** *The topological entropy of X_P coincides with the Mahler measure $m(P)$.*

The topological entropy of a $\mathbb{Z}^d$-action can be defined in at least five different ways, as outlined in [62, Section 13]. For instance, one has that

$$\begin{aligned} h(X_P) &= \lim_{\varepsilon\to 0} \limsup_{g(Q)\to+\infty} -\frac{1}{g(Q)} \log \lambda_{X_P}\left(\bigcap_{\mathbf{v}\in Q} \alpha_{\mathbf{v}}^{-1}(B_\varepsilon)\right) \\ &= \lim_{\varepsilon\to 0} \liminf_{g(Q)\to+\infty} -\frac{1}{g(Q)} \log \lambda_{X_P}\left(\bigcap_{\mathbf{v}\in Q} \alpha_{\mathbf{v}}^{-1}(B_\varepsilon)\right), \end{aligned} \tag{2.4}$$

where Q runs over all the rectangles $Q = \prod_{j=1}^n \{b_j, \dots, b_j + l_j - 1\}$, and $g(Q) := \min_{j=1,\dots,n} l_j$ denotes the *girth* of Q. Moreover, λ_{X_P} is the unique probability Haar measure on X_P, while B_ε denotes a ball centred at the identity of X_P and with radius ε. Such a ball can be taken with respect to any metric on X_P, but for the sake of convenience we are going to fix the distance

$$d_{\mathbb{T}^{\mathbb{Z}^n}}(x, y) := \sum_{\mathbf{v}\in\mathbb{Z}^n} 2^{-\|\mathbf{v}\|_\infty} \cdot d_{\mathbb{T}}(x_{\mathbf{v}}, y_{\mathbf{v}})$$

for every $x, y \in \mathbb{T}^{\mathbb{Z}^n}$, where $d_{\mathbb{T}}(e^{2\pi i t_1}, e^{2\pi i t_2}) := \operatorname{dist}(t_1 + \mathbb{Z}, t_2 + \mathbb{Z})$ denotes the standard distance on the unit circle, while $\|\mathbf{v}\|_\infty := \max\{|v_1|, \dots, |v_n|\}$ for every $\mathbf{v} = (v_1, \dots, v_n) \in \mathbb{Z}^n$. This choice of metric allows us to replace the balls B_ε appearing in (2.4) with balls inside only one copy of the unit circle, as the following result shows (see [62, Proposition 13.7]).

Lemma 2.5. *For every $P \in \mathbb{Z}[x_1^{\pm 1}, \dots, x_n^{\pm 1}]$ we have that*

$$\begin{aligned} h(X_P) &= \lim_{\varepsilon \to 0} \limsup_{g(Q) \to +\infty} -\frac{1}{g(Q)} \log \lambda_{X_P} \left(\bigcap_{\mathbf{v} \in Q} \alpha_{\mathbf{v}}^{-1}(B'_\varepsilon) \right) \\ &= \lim_{\varepsilon \to 0} \liminf_{g(Q) \to +\infty} -\frac{1}{g(Q)} \log \lambda_{X_P} \left(\bigcap_{\mathbf{v} \in Q} \alpha_{\mathbf{v}}^{-1}(B'_\varepsilon) \right), \end{aligned} \tag{2.6}$$

where $B'_\varepsilon := \{x \in \mathbb{T}^{\mathbb{Z}^n} \colon d_{\mathbb{T}}(x_{\mathbf{0}}, 1) < \varepsilon\}$.

Proof. Let $K_n := \sum_{\mathbf{v} \in \mathbb{Z}^n} 2^{-\|\mathbf{v}\|_\infty} \in \mathbb{R}$ and fix some $\varepsilon > 0$. Moreover, let $b_\varepsilon \in \mathbb{N}$ be any natural number such that $\sum_{\{\mathbf{v} \in \mathbb{Z}^n \colon \|\mathbf{v}\|_\infty > b_\varepsilon\}} 2^{-\|\mathbf{v}\|_\infty} < \varepsilon$, which surely exists. Then, for every pair of points $x, y \in \mathbb{T}^{\mathbb{Z}^n}$, if $d_{\mathbb{T}}(x_{\mathbf{v}}, y_{\mathbf{v}}) \leq \varepsilon$ for every $\mathbf{v} \in \mathbb{Z}^n$ with $\|\mathbf{v}\|_\infty \leq b_\varepsilon$, we have that $d_{\mathbb{T}^{\mathbb{Z}^d}}(x, y) \leq (K_n + 1)\varepsilon$. Hence, for every fixed $\varepsilon > 0$, and every rectangle $Q \subseteq \mathbb{Z}^n$ such that $g(Q) \geq b_\varepsilon$, we have that

$$\bigcap_{\mathbf{v} \in Q} \alpha_{\mathbf{v}}^{-1}(B_\varepsilon) \subseteq \bigcap_{\mathbf{v} \in Q} \alpha_{\mathbf{v}}^{-1}(B'_\varepsilon) \subseteq \bigcap_{\mathbf{v} \in Q} \alpha_{\mathbf{v}}^{-1}(B_{(K_n+1)\varepsilon}),$$

which allows us to conclude that the limits portrayed in (2.6) exist and coincide with $h(X_P)$. $\square$

The notions introduced so far are already sufficient to prove the equality between Mahler measures and entropies in the univariate case. To do so, we need a preliminary result about the entropy of linear maps.

Lemma 2.7. *Let $d \geq 1$ and $A \in \mathbb{C}^{d \times d}$ be a matrix with eigenvalues $\beta_1, \dots, \beta_d$. Then*

$$\lim_{N \to +\infty} \left(-\frac{1}{N} \log \lambda_{\mathbb{C}^d}(\{\underline{z}_d \in \mathbb{C}^d \colon \|A^k \underline{z}_d\|_\infty < 1,\ \forall k \in \{0, \dots, N-1\}\}) \right) = 2 \sum_{j=1}^{d} \log \max(|\beta_j|, 1),$$

where $\lambda_{\mathbb{C}^d}$ denotes Lebesgue's measure on $\mathbb{C}^d$.

Proof. Restricting A to the complement of its kernel, we can assume that A is nonsingular. Writing A in Jordan's normal form, we can decompose $\mathbb{C}^d$ as $\mathbb{C}^d = V_1 \oplus \cdots \oplus V_J$ in such a way that for every $j \in \{1, \dots, J\}$ the restriction of A to each eigenspace V_j has eigenvalues of a fixed modulus $r_j > 0$. Since all the norms on $\mathbb{C}^d$ are equivalent, we can choose a norm adapted to this decomposition. Therefore, if we define

$$B_j(N) := \{\underline{z} \in V_j \colon \|A_j^k \cdot \underline{z}\|_{V_j} < 1,\ \forall k \in \{0, \dots, N-1\}\}$$

for every $j \in \{1, \dots, J\}$, it suffices to show that

$$\liminf_{N \to +\infty} -\frac{\log \lambda_j(B_j(N))}{N} \geq 2 \dim(V_j) \log \max(r_j, 1) \tag{2.8}$$

$$\limsup_{N \to +\infty} -\frac{\log \lambda_j(B_j(N))}{N} \leq 2 \dim(V_j) \log \max(r_j, 1) \tag{2.9}$$

for every $j \in \{1, \dots, J\}$, where λ_j denote Lebesgue's measure on V_j. To prove the inequality (2.8), let us denote by $\mathbb{D}_j := \{\underline{z} \in V_j \colon \|\underline{z}\|_{V_j} < 1\}$ the unit disk of V_j. Then $\lambda_j(B_j(N)) \leq \lambda_j(\mathbb{D}_j)$ whenever $r_j \leq 1$, because in this case A_j noes not extend distances, whereas $\lambda_j(B_j(N)) \leq \lambda_j(A_j^{-(N-1)}(\mathbb{D}_j)) = |\det(A_j)|^{-2(N-1)} \lambda_j(\mathbb{D}_j)$ whenever $r_j > 1$. Therefore, if $r_j \leq 1$ we have that

$$\liminf_{N \to +\infty} -\frac{\log \lambda_j(B_j(N))}{N} \geq \liminf_{N \to +\infty} -\frac{\log \lambda_j(\mathbb{D}_j)}{N} = 0 = 2 \dim(V_j) \log \max(r_j, 1),$$

while if $r_j > 1$ we have that

$$\begin{aligned}\liminf_{N\to+\infty} -\frac{\log \lambda_j(B_j(N))}{N} &\geq \liminf_{N\to+\infty} \left(2\left(\frac{N-1}{N}\right)\log|\det(A_j)| - \frac{\log\lambda_j(\mathbb{D}_j)}{N}\right) = 2\log|\det(A_j)| \\ &= 2\dim(V_j)\log\max(r_j,1),\end{aligned}$$

which finally shows (2.8).

Now, to prove (2.9), we fix some $\delta > 0$ such that for every $j \in \{1,\dots,J\}$ we have that $r_j + \delta < 1$ whenever $r_j < 1$. Then, we define the norm

$$\|x\|_j' := \sum_{k=0}^{+\infty} \frac{\|A_j^k \cdot x\|_j}{(r_j+\delta)^k},$$

which is well defined thanks to the root test, because

$$\limsup_{k\to\infty}\left(\frac{\|A_j^k\cdot x\|_j}{(r_j+\delta)^k}\right)^{1/k} \leq \limsup_{k\to\infty}\frac{\|A_j^k\|^{1/k}\cdot\|x\|_j^{1/k}}{r_j+\delta} = \frac{r_j}{r_j+\delta} < 1,$$

and for the same reason is equivalent to $\|\cdot\|_j$, because $\|x\|_j \leq \|x\|_j' \leq C\cdot\|x\|_j$ where $C := \sum_{k=0}^{+\infty}\frac{\|A_j^k\|}{(r_j+\delta)^k}$. Note moreover that $A_j^{-k}(\mathbb{D}_j') \supseteq \{\underline{z} \in V_j \colon \|\underline{z}\|_j' < (r_j+\delta)^{-k}\}$ by construction, where $\mathbb{D}_j' := \{\underline{z} \in V_j \colon \|\underline{z}\|_j' < 1\}$ denotes the unit disk for the norm $\|\cdot\|_j'$. Therefore, if we define

$$B_j'(N) := \{\underline{z} \in V_j \colon \|A_j^k \underline{z}\|_j' < 1,\ \forall k \in \{0,\dots,N-1\}\},$$

we see that $B_j'(N) \supseteq \{\underline{z} \in V_j \colon \|\underline{z}\|_j' < (r_j+\delta)^{-(N-1)}\} \supseteq \{\underline{z} \in V_j \colon \|\underline{z}\|_j < C^{-1}(r_j+\delta)^{-(N-1)}\}$. In particular, we see that if $r_j < 1$ then $B_j'(N) \supseteq \mathbb{D}_j' \supseteq \mathbb{D}_{j,C}'$ where $\mathbb{D}_{j,C}' := \{\underline{z} \in V_j \colon \|\underline{z}\|_j < C^{-1}\}$. Therefore, we get that

$$\lambda_j(B_j'(N)) \geq \frac{\lambda_j(\mathbb{D}_{j,C}')}{(r_j+\delta)^{2\dim(V_j)(N-1)}}$$

whenever $r_j \geq 1$, whereas $\lambda_j(B_j'(N)) \geq \lambda_j(\mathbb{D}_{j,C}')$ whenever $r_j < 1$. This implies finally that

$$\begin{aligned}\limsup_{N\to+\infty} -\frac{\log\lambda_j(B_j(N))}{N} &= \limsup_{N\to+\infty} -\frac{\log\lambda_j(B_j'(N))}{N} \\ &\leq \limsup_{N\to+\infty} -\frac{\log\lambda_j(\mathbb{D}_{j,C})}{N} + 2\dim(V_j)\log\max(r_j+\delta,1)\left(\frac{N-1}{N}\right) \\ &= 2\dim(V_j)\log\max(r_j+\delta,1)\end{aligned}$$

for every $\delta > 0$ which is small enough, and this allows us to conclude by taking $\delta \to 0$. $\square$

Using the previous Lemma 2.7 we can prove the equality between Mahler measures and entropies in the univariate case.

Proposition 2.10. *Let $P \in \mathbb{Z}[x^{\pm 1}]$. Then $h(X_P) = m(P)$.*

Proof. We can assume without loss of generality that P has at least two non-zero coefficients. Indeed, if $P = 0$ then X_0 is isomorphic to the shift on $\mathbb{T}^{\mathbb{Z}}$, for which clearly $h(X_0) = \infty$, while if $P = cx^d$ is a monomial, then X_P is isomorphic to the shift on $(\mathbb{Z}/c\mathbb{Z})^{\mathbb{Z}}$, which implies that $h(X_{cx^d}) = \log|c|$.

Now, for every $N \geq 1$ we can consider the rectangle $Q_N = \{1, \dots, N\}$, to which we can associate the sets $B(N,\varepsilon) := \{x \in X_P \colon d_{\mathbb{T}}(x_j, 1) \leq \varepsilon, \ \forall j \in \{0, \dots, N-1\}\}$ for every $\varepsilon > 0$. When $0 < \varepsilon < \frac{1}{2}$, for every $N \geq 1$ and $x \in B(N,\varepsilon)$ there exists a unique $y = (y_0, \dots, y_{N-1}) \in \mathbb{R}^N$ such that $|y_j| < \varepsilon$ and $y_j \equiv x_j \pmod 1$ for every $j \in \{0, \dots, N-1\}$. This defines a map $\phi_N \colon B(N,\varepsilon) \to \mathbb{R}^N$. Now, write $P(x_1) = c_0 + \dots + c_d x_1^d$, and suppose that $c_0 c_d \neq 0$. Moreover, let $\beta_1, \dots, \beta_d$ be the roots of P, and $a_j := c_j / c_d$ for every $j \in \{1, \dots, d\}$. Then, if $N > d$ and $\varepsilon < (4 |c_d| \max(|\beta_j| + 1))^{-1}$, we have that

$$\phi_N(B(N,\varepsilon)) = \{(y_0, \dots, y_{N-1}) \in (-\varepsilon, \varepsilon)^n \colon c_0 y_k + \dots + c_d y_{k+d} = 0, \ \forall k \in \{0, \dots, N-d-1\}\},$$

which implies that each $\underline{y}_N = (y_0, \dots, y_{N-1}) \in \phi_N(B(N,\varepsilon))$ is determined by $\underline{y}_d = (y_0, \dots, y_{d-1})$, because $(y_k, \dots, y_{k+d-1}) = A^k \underline{y}_d$ for every $k \in \{0, \dots, N-d-1\}$, where

$$A := \begin{pmatrix} 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & 1 \\ -a_0 & -a_1 & -a_2 & \dots & -a_{d-1} \end{pmatrix}$$

is the companion matrix of P. Therefore, the projection of $B(N,\varepsilon)$ on the first d coordinates is injective, and its image coincides with the set $\mathfrak{B}_{\mathbb{R}^d}(N,\varepsilon) := \{\underline{t}_d \in \mathbb{R}^d \colon \|A^k \underline{t}_d\|_\infty < \varepsilon, \ \forall k \in \{0, \dots, N-d-1\}\}$ which has the same Lebesgue measure as $B(N,\varepsilon)$.

This implies that the computation of the entropy $h(X_P)$ is essentially reduced to the computation of the entropy of the action of A on $\mathbb{R}^d$. More precisely, we have that

$$\frac{\lambda_{X_P}(B(N,\varepsilon))}{\lambda_{X_P}(B(N+1,\varepsilon))} = |c_d| \frac{\lambda_{\mathbb{R}^d}(\mathfrak{B}_{\mathbb{R}^d}(N,\varepsilon))}{\lambda_{\mathbb{R}^d}(\mathfrak{B}_{\mathbb{R}^d}(N+1,\varepsilon))},$$

for every $N \geq d$, which implies that $\lambda_{X_P}(B(N,\varepsilon)) = |c_d|^{N-d} \lambda_{\mathbb{R}^d}(\mathfrak{B}_{\mathbb{R}^d}(N,\varepsilon))$ for every $N \geq d$. Moreover

$$\lambda_{\mathbb{R}^d}(\mathfrak{B}_{\mathbb{R}^d}(N,\varepsilon))^2 \geq \lambda_{\mathbb{C}^d}(\mathfrak{B}_{\mathbb{C}^d}(N,\varepsilon)) \geq \lambda_{\mathbb{R}^d}\left(\mathfrak{B}_{\mathbb{R}^d}\left(N, \frac{\varepsilon}{2}\right)\right)^2,$$

where $\mathfrak{B}_{\mathbb{C}^d}(N,\varepsilon) := \{\underline{z}_d \in \mathbb{C}^d \colon \|A^k \underline{z}_d\|_\infty < \varepsilon, \ \forall k \in \{0, \dots, N-d-1\}\}$. Hence, we see that

$$h(X_P) = \lim_{\varepsilon \to 0} \limsup_{N \to +\infty} -\frac{1}{N} \log \lambda_{X_P}(B(N,\varepsilon)) = \log|c_d| + \lim_{\varepsilon \to 0} \limsup_{N \to +\infty} \left(-\frac{1}{2N} \log \lambda_{\mathbb{C}^d}(\mathfrak{B}_{\mathbb{C}^d}(N,\varepsilon))\right),$$

which allows us to conclude using Lemma 2.7. □

To conclude the proof of the equality between the entropy of X_P and the Mahler measure $m(P)$ we are going to observe that each dynamical system X_P can be "approximated" by other dynamical systems X_{P_A} associated to polynomials P_A which depend on a smaller number of variables. More precisely, for every matrix $A = (a_{i,j}) \in \mathbb{Z}^{m \times n}$ and every Laurent polynomial $P \in \mathbb{Z}[x_1^{\pm 1}, \dots, x_n^{\pm 1}] \setminus \{0\}$, we let

$$P_A(x_1, \dots, x_m) := P(x_1^{a_{1,1}} \cdots x_m^{a_{m,1}}, \dots, x_1^{a_{1,n}} \cdots x_m^{a_{m,n}}), \tag{2.11}$$

which defines a polynomial $P_A \in \mathbb{Z}[x_1^{\pm 1}, \dots, x_m^{\pm 1}]$. Moreover, for every matrix A we define the invariant

$$\rho(A) := \min\{\|v\|_\infty \colon v \in \ker(A) \cap (\mathbb{Z}^n \setminus \{0\})\},$$

which measures how "sparse" the lattice $\ker(A) \cap \mathbb{Z}^n$ is inside the vector space $\ker(A) \subseteq \mathbb{R}^n$. Then, it turns out that the entropy of the dynamical system X_P can be approximated by the entropies of the dynamical systems X_{P_A} when $\rho(A) \to +\infty$, as the following theorem shows.

Theorem 2.12. *For every Laurent polynomial $P \in \mathbb{Z}[x_1^{\pm 1}, \dots, x_n^{\pm 1}]$ we have that $\lim_{\rho(A)\to+\infty} h(X_{P_A}) = h(X_P)$.*

Proof. Let us just sketch the main idea. First of all, one can observe that $h(X_{P_A}) = h(X_P)$ whenever we have that $\operatorname{rk}(A) \geq n$. Therefore, up to multiplying the matrix A on the right by a matrix $B \in \mathbb{Z}^{n\times n}$ such that $\det(B) \neq 0$, we can assume that $\operatorname{rk}(A) = m$ and that the linear map $A\colon \mathbb{Z}^n \to \mathbb{Z}^m$ is surjective, as explained for instance in . Then, let $\psi_A \colon \mathbb{T}^{\mathbb{Z}^m} \to \mathbb{T}^{\mathbb{Z}^n}$ be the map defined by $\psi_A((x_{\mathbf{w}})_{\mathbf{w}\in\mathbb{Z}^m}) = (x_{A\cdot\mathbf{v}})_{\mathbf{v}\in\mathbb{Z}^n}$, and observe that

$$\psi_A(\mathbb{T}^{\mathbb{Z}^m}) = \{(x_{\mathbf{v}}) \in \mathbb{T}^{\mathbb{Z}^n} : x_{\mathbf{v}} = x_{\mathbf{v}+\mathbf{k}},\ \forall \mathbf{k} \in \ker(A) \cap \mathbb{Z}^n\},$$

and that $\psi_A(X_{P_A}) = X_P \cap \psi_A(\mathbb{T}^{\mathbb{Z}^m})$. Now, the main idea is to prove the weak convergence of measures $(\psi_A)_*(\lambda_{X_{P_A}}) \to \lambda_{X_P}$ as $\rho(A) \to +\infty$, and to show that this weak convergence is strong enough to imply the convergence of entropies. □

Remark 2.13. The original proof provided by Lind, Schmidt and Ward [48], which is also surveyed in [62, Theorem 18.1], uses the approximation provided by Theorem 2.12 only to prove that $h(X_P) \leq m(P)$, while it uses another argument, based on Riemann sums, in order to prove the reverse inequality.

Given a Laurent polynomial $P \in \mathbb{Z}[x_1^{\pm 1}, \dots, x_n^{\pm 1}]$, we can apply the previous Theorem 2.12 to the sequence of matrices $A_d := (1, d, d^2, \dots, d^{n-1}) \in \mathbb{Z}^{1\times n}$, for which $\rho(A_d) = d$ and $P_{A_d}(x_1) = P(x_1, x_1^d, \dots, x_1^{d^{n-1}})$. Therefore, thanks to Proposition 2.10 and Theorem 2.12, we see that

$$h(X_P) = \lim_{d\to+\infty} h(X_{P_{A_d}}) = \lim_{d\to+\infty} m(P_{A_d}).$$

To conclude, it is therefore sufficient to prove that the Mahler measure $m(P)$ can be approximated by the univariate Mahler measures $m(P_{A_d})$, or more generally by the Mahler measures $m(P_A)$ for $\rho(A) \to +\infty$. The first result is due to Lawton [44], while its generalization to arbitrary sequences of matrices is due to a joint work between Brunault, Guilloux, Mehrabdollahei and the author of the present notes [12].

Theorem 2.14 (Brunault, Guilloux, Mehrabdollahei, P.). *For every $P \in \mathbb{C}[x_1^{\pm 1}, \dots, x_n^{\pm 1}] \setminus \{0\}$, we have that $m(P_A) \to m(P)$ as $\rho(A) \to +\infty$. More precisely, if P has $k_P \geq 2$ non-zero coefficients then for every matrix $A \in \mathbb{Z}^{m\times n}$ such that $\rho(A) \geq \rho_0(P) := \max(7 \cdot \operatorname{diam}(N_P)^2, \exp(2(k_P - 1)\max(n, 5)))$ we have that*

$$|m(P_A) - m(P)| \leq 8 \cdot (36 \cdot e \cdot k_P)^{n-1} \cdot \operatorname{diam}(N_P)^{\frac{1}{k_P-1}} \cdot \log(\rho(A))^n \cdot \rho(A)^{-\frac{1}{k_P-1}},$$

while $m(P) = m(P_A)$ for every $A \in \mathbb{Z}^{m\times n}$ if P has one or less non-zero coefficients.

Proof. Let us give a sketch of the proof. If $Q \in \mathbb{C}[x_1^{\pm 1}, \dots, x_n^{\pm 1}]$ does not vanish on $\mathbb{T}^n$, we in fact have the much better upper bound

$$|m(Q_A) - m(Q)| \leq (n+1) \cdot 3^n \cdot \max_{\mathcal{C}_\delta} |f_Q| \cdot e^{-\delta\rho(A)}, \tag{2.15}$$

for every $\delta > 0$ such that Q does not vanish on $\mathcal{C}_\delta := \{\underline{x}_n \in (\mathbb{C}^\times)^n : \sum_{j=1}^n |\log|x_j|| \leq \delta\}$, where f_Q denotes the unique holomorphic function on $\mathcal{C}_\delta$ such that $df_Q = \frac{1}{2} d(QQ^*)/(QQ^*)$ and $f_Q = \log|Q|$ on $\mathbb{T}^n$. To obtain this result, we write

$$|m(Q_A) - m(Q)| = \left| \int_{\mathbb{T}^n} f_Q d\mu_A - \int_{\mathbb{T}^n} f_Q d\mu_n \right| = \left| \sum_{v\in\ker(A)\cap(\mathbb{Z}^n\setminus\{0\})} c_v(f_Q) \right| \leq \sum_{v\in\ker(A)\cap(\mathbb{Z}^n\setminus\{0\})} |c_v(f_Q)| \tag{2.16}$$

where μ_A denotes the push-forward of μ_m under the natural map $\mathbb{T}^m \to \mathbb{T}^n$ induced by the transpose of A, while $c_v(f_Q) := \int_{\mathbb{T}^n} f_Q(\underline{x}_n)\underline{x}_n^{-v} d\mu_n$ denotes the v-th Fourier coefficient of f_Q. Since f_Q is holomorphic, these Fourier coefficients decay exponentially. More precisely, we can deform the integration set $\mathbb{T}^n$ to the boundary of the region of holomorphicity $\mathcal{C}_\delta$, and if we do so we get $|c_v(f_Q)| \leq \max_{\mathcal{C}_\delta}|f_Q| \cdot e^{-\delta\|v\|_\infty}$. Combining this estimate with (2.16), and setting $d_A := \dim(\ker(A))$, we see that

$$|m(Q_A) - m(Q)| \leq \max_{\mathcal{C}_\delta}|f_Q| \cdot \sum_{v \in \ker(A) \cap (\mathbb{Z}^n \setminus \{0\})} e^{-\delta\|v\|_\infty} \leq \max_{\mathcal{C}_\delta}|f_Q| \cdot (d_A + 1) \cdot 3^{d_A} \cdot e^{-\delta\rho(A)},$$

where the last inequality follows from Abel's summation formula, and immediately implies (2.15).

To treat the general case, we define $Q_\varepsilon := PP^* + \varepsilon$ for every $\varepsilon > 0$, where $P^* := \sum_{v \in \mathbb{Z}^n} \overline{c_v(P)} \underline{x}_n^{-v}$ is the conjugate reciprocal of P. Then, thanks to the continuity of the Mahler measure as a function of the coefficients of a family of polynomials with fixed Newton polytope, explained in Remark 2.2, we see that $m(Q_\varepsilon) \to 2m(P)$ and $m((Q_\varepsilon)_A) \to 2m(P_A)$ as $\varepsilon \to 0$. In particular, for every $\varepsilon > 0$ we can write

$$|m(P_A) - m(P)| \leq \left| m(P_A) - \frac{m((Q_\varepsilon)_A)}{2} \right| + \left| \frac{m((Q_\varepsilon)_A) - m(Q_\varepsilon)}{2} \right| + \left| \frac{m(Q_\varepsilon)}{2} - m(P) \right|, \tag{2.17}$$

and $|m((Q_\varepsilon)_A) - m(Q_\varepsilon)| \to 0$ whenever $\rho(A) \to +\infty$, because $PP^*(\mathbb{T}^n) \subseteq \mathbb{R}_{\geq 0}$ and thus Q_ε does not vanish on $\mathbb{T}^n$. Moreover, one can show that

$$\frac{m(Q_\varepsilon)}{2} - m(P) = \int_0^{+\infty} \mu_n\left(\left\{\xi \in \mathbb{T}^n \colon |P(\xi)|^2 \leq \frac{\varepsilon}{e^{2t} - 1}\right\}\right) dt$$

using some standard measure theory, and that for every $\alpha \in (0,1)$ and $r > 0$ we have that

$$\mu_n(\{\xi \in \mathbb{T}^n \colon |P(\xi)| \leq r\}) \leq 6 \cdot k_P \cdot (18 \cdot n \cdot k_P^2)^{n-1} \cdot \alpha^{1-n} \cdot \left(\frac{r}{L_\infty(P)}\right)^{\frac{1-\alpha}{k_P - 1}}, \tag{2.18}$$

as can be proved using a technique introduced by Dobrowolski [25, Theorem 1.4] and Dimitrov and Habegger [23, Lemma A.3], where we write $L_\infty(P) := \max_v |c_v(P)|$. Therefore

$$0 \leq \frac{m(Q_\varepsilon)}{2} - m(P) \leq 12 \cdot k_P^2 \cdot (18 \cdot n \cdot k_P^2)^{n-1} \cdot \frac{\alpha^{1-n}}{1-\alpha} \cdot \varepsilon^{\frac{1-\alpha}{2(k_P-1)}}, \tag{2.19}$$

for every $\alpha \in (0,1)$. Finally, one can prove by an explicit computation that the function $f_\varepsilon := \log(Q_\varepsilon)$ is holomorphic on the set $\mathcal{C}_{\delta(\varepsilon)}$, and that

$$\max_{\mathcal{C}_{\delta(\varepsilon)}}|f_\varepsilon| \leq |\log(\varepsilon)| + 2|\log(L_1(P))| + 3, \tag{2.20}$$

where $\delta(\varepsilon) := \min\left(\frac{\log(4/3)}{d_P}, \frac{\sqrt{\varepsilon}}{d_P L_1(P)}\right)$ and $L_1(P) := \sum_{v \in \mathbb{Z}^n} |c_v(P)|$, while $d_P := \operatorname{diam}(N_P)$ is the diameter of the Newton polytope of P. To conclude, one can combine (2.16) with (2.20) to get an effective upper bound for $|m((Q_\varepsilon)_A) - m(Q_\varepsilon)|$, and one can plug such an effective upper bound in (2.17), together with (2.19), to obtain an effective upper bound for $|m(P_A) - m(P)|$ which depends on two parameters α and ε. It turns out that this upper bound is minimized by choosing $\varepsilon = \left(\frac{(1-\alpha) \cdot L_1(P) \cdot d_P \cdot \log(\rho(A))}{(k_P - 1) \cdot \rho(A)}\right)^2$ and $\alpha = \frac{n(k_P - 1)}{\log(\rho(A))}$, which allows us to conclude our proof. □

Let us conclude this lecture with some remarks. First of all, let us observe that the relation between entropies and Mahler measures can be generalized to compute the entropy of any $\mathbb{Z}^n$-action α on a compact abelian group X. More precisely, the dual group $\hat{X}$ is a module over the ring $\mathbb{Z}[x_1^{\pm 1}, \dots, x_n^{\pm 1}]$ thanks to the action $P * \hat{x} := \sum_{v \in \mathbb{Z}^n} c_v(P) \cdot \hat{\alpha}_v(\hat{x})$, where $\hat{\alpha}$ denotes the $\mathbb{Z}^n$ action on $\hat{X}$ induced by α. Vice-versa, if M is a countable $\mathbb{Z}[x_1^{\pm 1}, \dots, x_n^{\pm 1}]$-module, then $\mathbb{Z}^n$ acts on M by setting $v * m := \underline{x}_n^v \cdot m$, and we can consider the dual action on $\hat{M}$, which is a compact abelian group. Now, starting from a $\mathbb{Z}^n$-action on a compact abelian group X, and letting $M := \hat{X}$, we can filter M by a sequence of Noetherian $\mathbb{Z}[x_1^{\pm 1}, \dots, x_n^{\pm 1}]$-modules M_k, with dual dynamical systems X_k, such that $h(X_k) \to h(X)$ as $k \to +\infty$. Moreover, if M is a Noetherian $\mathbb{Z}[x_1^{\pm 1}, \dots, x_n^{\pm 1}]$-module, then M admits a *decomposition series* $\{0\} = M_0 \subseteq M_1 \subseteq \cdots \subseteq M_s = M$ such that $M_j / M_{j-1} \cong \mathbb{Z}[x_1^{\pm 1}, \dots, x_n^{\pm 1}]/\mathfrak{p}_j$, for some prime ideal $\mathfrak{p}_j \subseteq \mathbb{Z}[x_1^{\pm 1}, \dots, x_n^{\pm 1}]$. Then, thanks to an addition formula due to Yuzvinskii (see [62, Theorem 14.1]), one can prove that

$$h(\hat{M}) = h(\widehat{M}_{s-1}) + h(X_{\mathfrak{p}_s}) = \cdots = \sum_{j=1}^{r} h(X_{\mathfrak{p}_j}),$$

where for every ideal $I \subseteq \mathbb{Z}[x_1^{\pm 1}, \dots, x_n^{\pm 1}]$ we let X_I denote the dynamical system associated to the module $\mathbb{Z}[x_1^{\pm 1}, \dots, x_n^{\pm 1}]/I$. Finally, for every prime ideal $\mathfrak{p} \subseteq \mathbb{Z}[x_1^{\pm 1}, \dots, x_n^{\pm 1}]$, we have that $h(X_{\mathfrak{p}}) = m(P)$ if $\mathfrak{p}$ is generated by P, thanks to Theorem 2.3. On the other hand, $h(X_{\mathfrak{p}}) = 0$ if $\mathfrak{p}$ is not principal. Indeed, fix $P \in \mathfrak{p} \setminus \{0\}$ and $Q \in \mathfrak{p} \setminus (P)$, and observe that

$$h(X_P) = h(X_P) + h(X_{(P,Q)}), \tag{2.21}$$

thanks again to Yuzvinskii's addition formula, and to the fact that the sequence

$$0 \to \mathbb{Z}[x_1^{\pm 1}, \dots, x_n^{\pm 1}]/(P) \xrightarrow{\cdot Q} \mathbb{Z}[x_1^{\pm 1}, \dots, x_n^{\pm 1}]/(P) \to \mathbb{Z}[x_1^{\pm 1}, \dots, x_n^{\pm 1}]/(P,Q) \to 0$$

is exact. Since $h(X_P) = m(P) \in \mathbb{R}$, the identity (2.21) necessarily implies that the entropy of $X_{(P,Q)}$ vanishes. To conclude, one can proceed by induction on the number of generators of $\mathfrak{p}$.

This shows in particular that the entropy of a $\mathbb{Z}^n$-action on a compact abelian group X is either infinite or lies in the closure of the countable semigroup $\mathcal{M}_\infty := \bigcup_{n=1}^{+\infty} m(\mathbb{Z}[x_1^{\pm 1}, \dots, x_n^{\pm 1}] \setminus \{0\})$. This set $\mathcal{M}_\infty$ bears some similarities with the set of *Pisot numbers* S, *i.e.* those real algebraic integers $\lambda > 1$ whose conjugates are all smaller than one in absolute value. This set S was shown to be closed by Salem [59]. Based on this result, Boyd [6] proposed the following conjecture.

Conjecture 2.22 (Boyd)**.** *The set* $\mathcal{M}_\infty := \bigcup_{n=1}^{+\infty} m(\mathbb{Z}[x_1^{\pm 1}, \dots, x_n^{\pm 1}] \setminus \{0\})$ *is closed.*

It is not difficult to show that, if this conjecture was true, then the first part of Question 1.8 would have a positive answer. More precisely, if there existed a sequence of polynomials $\{P_k\}_{k=1}^{+\infty} \subseteq \mathbb{Z}[x_1] \setminus \{0\}$ such that $m(P_k) > 0$ for every $k \geq 1$, and $m(P_k) \to 0$ as $k \to +\infty$, the closure of $\mathcal{M}_\infty$ would be $\mathbb{R}_{\geq 0}$, because we would have that

$$\alpha = \lim_{k \to +\infty} \left\lfloor \frac{\alpha}{m(P_k)} \right\rfloor \cdot m(P_k) = \lim_{k \to +\infty} m\left(P_k^{\left\lfloor \frac{\alpha}{m(P_k)} \right\rfloor} \right)$$

for every $\alpha \geq 0$. However, this contradicts Conjecture 2.22 because $\mathcal{M}_\infty$ is countable, and therefore does not coincide with $\mathbb{R}_{\geq 0}$. To conclude, let us mention a recent conjecture on the topological nature of $\mathcal{M}_\infty$, proposed by Smyth [50, Section 2.1].

Conjecture 2.23 (Smyth)**.** *The set* $\mathcal{M}_\infty := \bigcup_{n=1}^{+\infty} m(\mathbb{Z}[x_1^{\pm 1}, \dots, x_n^{\pm 1}] \setminus \{0\})$ *is a* Thue set, *which means that:*

- $\mathcal{M}_\infty$ *is closed;*

- *for every $n \geq 0$ the $(n+1)$-st derived subset $\mathcal{M}_\infty^{(n+1)}$ is non-empty, and each of its elements is a limit from both sides of elements of the n-th derived subset $\mathcal{M}_\infty^{(n)}$;*
- $\min(\mathcal{M}_\infty^{(n)}) \to +\infty$ *as $n \to +\infty$.*

Moreover, for each $n \geq 1$ we have that $\mathcal{M}_\infty^{(n)} = \mathcal{M}_{n+1} \setminus \mathcal{M}_n$, where $\mathcal{M}_n := m(\mathbb{Z}[x_1^{\pm 1}, \dots, x_n^{\pm 1}] \setminus \{0\})$.

3 Graphs and links

We have seen that Mahler measures can be used to detect the distribution of roots of univariate polynomials, and also to compute the entropy of essentially every $\mathbb{Z}^n$-action on a metrizable compact abelian group. In this lecture, we will see how Mahler measures, and in particular Lehmer's question, appear in low dimensional topology and combinatorics.

non-Archimedean growth of Pierce-Lehmer sequences

To witness this appearance, we will relate certain combinatorial and topological invariants to the Pierce-Lehmer sequence $\{\Delta_N(P)\}_{N=1}^{+\infty} \subseteq \mathbb{Z}$ associated to a polynomial $P \in \mathbb{Z}[x]$, defined in (1.4). We know from Theorem 1.6 that the Archimedean growth of Pierce-Lehmer sequences is detected by the Mahler measure of P. A similar result turns out to be true in a non-Archimedean setting.

More precisely, suppose that we want to study the prime factorization of the elements $\Delta_N(P)$ associated to the Pierce-Lehmer sequence of P. More precisely, suppose that we want to know how the p-adic valuation $\mathrm{ord}_p(\Delta_N(P))$ varies, where p is a fixed prime number. It turns out that this invariant grows at most linearly as a function of $N \to +\infty$, and that the growth coefficient can be expressed in terms of the *p-adic Mahler measure* of the polynomial P. In fact, one can derive a completely explicit expression for $\mathrm{ord}_p(\Delta_N(P))$ by studying the p-adic behavior of the roots of P. To be more precise, we need to introduce some notation.

Notation 3.1. Given a prime number p, we let $\mathbb{C}_p$ be the completion of the algebraic closure $\overline{\mathbb{Q}}_p$ of the field of p-adic numbers $\mathbb{Q}_p$, and we let $\mathrm{ord}_p(\cdot)$ and $|\cdot|_p := p^{-\mathrm{ord}_p(\cdot)}$ denote the p-adic valuation and absolute values defined on $\mathbb{C}_p$, which we normalize by imposing that $\mathrm{ord}_p(p) = 1$. Moreover, we let $\mathcal{O}_p$ denote the ring of integers of $\overline{\mathbb{Q}}_p$, and $\mathfrak{m}_p$ denote its maximal ideal. Then, we have a reduction map $\pi_p \colon \mathcal{O}_p \twoheadrightarrow \mathcal{O}_p/\mathfrak{m}_p \cong \overline{\mathbb{F}}_p$, and a *Teichmüller lift* $\tau_p \colon \overline{\mathbb{F}}_p^\times \to \mathcal{O}_p^\times$ which is an injective group homomorphism. More precisely, given $x \in \overline{\mathbb{F}}_p^\times$ we define $\tau_p(x) \in \mathcal{O}_p^\times$ to be the unique element such that $\pi_p(\tau_p(x)) = x$ and $\tau_p(x)^M = 1$, where M is the multiplicative order of x in $\overline{\mathbb{F}}_p^\times$. Note that such an element $\tau_p(x)$ exists by Hensel's lemma.

Since the multiplicative order of every $\xi \in \overline{\mathbb{F}}_p^\times$ is finite and coprime to p, we see that $\tau_p(\overline{\mathbb{F}}_p^\times)$ consists of roots of unity whose order is coprime to p. Moreover, one can see that $\mathrm{ord}_p(\beta - \tau_p(\pi_p(\beta))) > 0$ for every $\beta \in \mathcal{O}_p^\times$. Therefore, for every $\beta \in \mathcal{O}_p^\times$ and every $N \in \mathbb{N}$ the quantity

$$r_{p,N}(\beta) := \min(\{\mathrm{ord}_p(N)\} \cup \{s \in \mathbb{N} \colon p^s(p-1)\mathrm{ord}_p(\beta - \tau_p(\pi_p(\beta))) \geq 1\})$$

is finite. Finally, given a polynomial $P = a_0 + \dots + a_d x^d = a_d(x - \alpha_1) \cdots (x - \alpha_d) \in \mathbb{C}_p[x]$, we define the *$p$-adic Mahler measure* of P to be the real number

$$m_p(P) := \log|a_d|_p + \sum_{j=1}^{d} \log\max(|\alpha_j|_p, 1), \tag{3.2}$$

which can also be computed thanks to the simpler formula $m_p(P) = \log\max(|a_0|_p, \dots, |a_d|_p)$.

Using the invariants introduced in Notation 3.1, we can relate the p-adic valuations of Pierce-Lehmer sequences to the p-adic behaviour of the roots of P, thanks to the following result, which we have proven in joint work with Vallières [52], inspired by an idea due to Ueki [73].

Theorem 3.3 (Ueki, P. & Vallières). *Let $P \in \mathbb{Z}[t] \setminus \{0\}$ be a polynomial which does not vanish on roots of unity. Moreover, let $p \in \mathbb{N}$ be a prime number, and let $N \geq 1$ be an integer. Then, the p-adic valuation of the Pierce-Lehmer sequence $\Delta_N(P)$ defined in (1.4) can be expressed as*

$$\operatorname{ord}_p(\Delta_N(P)) = \mu_p(P) \cdot N + \#B_{p,N}(P) \cdot \operatorname{ord}_p(N) + \nu_{p,N}(P),$$

where $\mu_p(P) := -m_p(P)/\log(p)$ and $B_{p,N}(P) := \{\beta \in \overline{\mathbb{Q}}_p \colon P(\beta) = 0,\ |\beta|_p = 1,\ |\beta^N - 1|_p < 1\}$, while

$$\nu_{p,N}(F) := \sum_{\beta \in B_{p,N}(F)} \left(\operatorname{ord}_p(\beta^{p^{r_{p,N}(\beta)}} - \tau_p(\pi_p(\beta))^{p^{r_{p,N}(\beta)}}) - r_{p,N}(\beta) \right).$$

Proof. For every $\alpha \in \mathbb{C}_p$, one has that $|\alpha^N - 1|_p = |\alpha|_p^N$ if $|\alpha|_p > 1$, and $|\alpha^N - 1|_p = 0$ if $|\alpha|_p < 1$. Therefore, if we write $P(t) = a_d(t - \alpha_1) \cdots (t - \alpha_d)$, we see from the definition of Pierce-Lehmer sequence (1.4) and of p-adic Mahler measure (3.2) that

$$\begin{aligned} |\Delta_N(F)|_p = |a_d|_p^N \cdot \prod_{j=1}^{d} |\alpha_j^N - 1|_p &= |a_d|_p^N \cdot \left(\prod_{\substack{j=1,\dots,d \\ |\alpha_j|_p > 1}} |\alpha_j|_p^N \right) \cdot \left(\prod_{\substack{j=1,\dots,d \\ |\alpha_j|_p = 1}} |\alpha_j^N - 1|_p \right) \\ &= \exp(m_p(F))^N \cdot \prod_{\substack{j=1,\dots,d \\ |\alpha_j|_p = 1}} |\alpha_j^N - 1|_p, \end{aligned}$$

which reduces our problem to the determination of the p-adic valuations $|\alpha^N - 1|_p$ associated to a number $\alpha \in \overline{\mathbb{Q}}_p$ such that $|\alpha|_p = 1$, which is equivalent to say that $\alpha \in \mathcal{O}_p^\times$.

To do so, suppose first of all that N is not a multiple of the multiplicative order of $\pi_p(\alpha)$, and write $\xi := \tau_p(\pi_p(\alpha))$. Then, one easily sees that for every root of unity $\zeta \in \mathcal{O}_p^\times$ whose order divides N, the order of ζ/ξ is not a power of p, which implies that $|\xi - \zeta|_p = |1 - \zeta/\xi|_p = 1$. Therefore, we see that for every $\alpha \in \mathcal{O}_p^\times$ such that the multiplicative order of $\pi_p(\alpha)$ is not a multiple of N we have that

$$|\alpha^N - 1|_p = \prod_{\substack{\zeta \in \mathcal{O}_p^\times \\ \zeta^N = 1}} |\alpha - \zeta|_p = \prod_{\substack{\zeta \in \mathcal{O}_p^\times \\ \zeta^N = 1}} |\alpha - \xi + \xi - \zeta|_p = 1.$$

Suppose on the other hand that N is a multiple of the multiplicative order of $\pi_p(\alpha)$, which we denote by M, and let $m := \operatorname{ord}_p(M)$. Moreover, if we let $\zeta \in \mathcal{O}_p$ be any root of unity whose multiplicative order divides N, and we denote again $\xi := \tau_p(\pi_p(\alpha))$, we see that $|\zeta - \xi|_p = 1$ unless the multiplicative order of $\mu := \zeta/\xi$ is a power of p. In this second case, if the multiplicative order of μ is p^s and $s \geq r$, where $r := r_{p,N}(\alpha)$ is the invariant defined in Notation 3.1, we see that $\operatorname{ord}_p(\zeta - \xi) = \operatorname{ord}_p(1 - \mu) = p^{1-s}(p-1)^{-1} < \operatorname{ord}_p(\alpha - \xi)$. Putting all this information together, we see that for every $\alpha \in \mathcal{O}_p^\times$ such that the multiplicative order of

$\pi_p(\alpha)$ is a multiple of N, we have that

$$
\begin{aligned}
|\alpha^N - 1|_p &= \prod_{\zeta \in W_N} |\alpha - \zeta|_p = \prod_{\mu \in W_{p^m}} |\alpha - \xi\mu|_p = |\alpha - \xi|_p \cdot \left(\prod_{\mu \in W_{p^r} \setminus \{1\}} |\alpha - \xi\mu|_p \right) \cdot \left(\prod_{\mu \in W_{p^m} \setminus W_{p^r}} |1 - \mu|_p \right) \\
&= |\alpha - \xi|_p \cdot \left(\prod_{\mu \in W_{p^r} \setminus \{1\}} \frac{|\alpha - \xi\mu|_p}{|1 - \mu|_p} \right) \cdot \left(\prod_{\mu \in W_{p^m} \setminus \{1\}} |1 - \mu|_p \right) \\
&= |\alpha - \xi|_p \cdot \left(\prod_{\mu \in W_{p^r} \setminus \{1\}} \frac{|\alpha - \xi\mu|_p}{|1 - \mu|_p} \right) \cdot |N|_p
\end{aligned}
$$

where $m = \mathrm{ord}_p(N)$ and $r := r_{p,N}(\alpha)$, while for every $k \geq 1$ we let $W_k \subseteq \mathcal{O}_p^\times$ denote the group of roots of unity whose order divides k. Therefore, we see that

$$
\begin{aligned}
\mathrm{ord}_p(\alpha^N - 1) &= \mathrm{ord}_p(N) + \mathrm{ord}_p(\alpha - \xi) + \sum_{\mu \in W_{p^r} \setminus \{1\}} \left(\mathrm{ord}_p(\alpha - \xi\mu) - \mathrm{ord}_p(1 - \mu) \right) \\
&= \mathrm{ord}_p(\alpha^{p^r} - \xi^{p^r}) - r,
\end{aligned}
$$

which allows us to conclude. □

Spanning trees and Pierce-Lehmer sequences

The previous Theorem 3.3 can be used to determine the growth rate of the number of *spanning trees* in a tower of finite graphs. To state this result, we introduce some elements of graph theory.

First of all, an *undirected graph* X is given by a set of *vertices* V_X, a set of *directed edges* $\mathbf{E}_X$, two maps $o, t \colon \mathbf{E}_X \to V_X$, called *origin* and *terminus*, and an *inversion map* $\iota \colon \mathbf{E}_X \to \mathbf{E}_X$ such that $o(\iota(e)) = t(e)$ and $t(\iota(e)) = o(e)$. A map of graphs $f \colon Y \to X$ is given by two maps $V_Y \to V_X$ and $\mathbf{E}_Y \to \mathbf{E}_X$, both denoted again by f, such that $(o(f(e)), t(f(e))) = (f(o(e)), f(t(e)))$ and $f(\iota(e)) = \iota(f(e))$ for every $e \in \mathbf{E}_Y$. Then, we say that f is a *cover* if $f \colon V_Y \to V_X$ is surjective, and for every $v \in V_Y$ the restriction of f to the *star of edges* $\mathbf{E}_{Y,v} := \{e \in \mathbf{E}_Y \colon o(e) = v\}$ induces a bijection $f|_{\mathbf{E}_{Y,v}} \colon \mathbf{E}_{Y,v} \xrightarrow{\sim} \mathbf{E}_{X,f(v)}$. Moreover, f is a *Galois cover* if Y is connected and if for every $v \in V_X$ the group $\mathrm{Aut}(Y/X) := \{\sigma \in \mathrm{Aut}(Y) \colon f \circ \sigma = \sigma\}$ acts transitively on the fiber $f^{-1}(v)$. In this case, we define the Galois group of the cover $Y \to X$ as $\mathrm{Gal}(Y/X) := \mathrm{Aut}(Y/X)$.

A concrete way to construct a Galois cover $Y \to X$ is to start from a *voltage assignment* α on the graph X, which is a map $\alpha \colon \mathbf{E}_X \to G$ with values in a group G, such that $\alpha(\iota(e)) = \alpha(e)^{-1}$ for each $e \in \mathbf{E}_X$. Indeed, to such a voltage assignment α one can associate a new graph $X(G, \alpha)$, by setting $V_{X(G,\alpha)} := V_X \times G$ and $\mathbf{E}_{X(G,\alpha)} := \mathbf{E}_X \times G$, such that $o(e, g) := (o(e), g)$ and $t(e, g) := (t(e), g \cdot \alpha(e))$, while $\iota(e, g) := (\iota(e), g \cdot \alpha(e))$ for every $(e, g) \in \mathbf{E}_{X(G,\alpha)}$. It is easy to see that the canonical projection $X(G, \alpha) \to X$, defined on vertices by $(v, g) \mapsto v$ and on edges by $(e, g) \mapsto e$, is a cover of graphs. This cover is also Galois whenever $X(G, \alpha)$ is connected, and one can conversely check that every finite Galois cover arises in this way. Therefore, given a finite graph X and a voltage assignment $\alpha \colon \mathbf{E}_X \to G$, we get an induced system of finite voltage assignments $\alpha_H \colon \mathbf{E}_X \to G/H$, indexed on the normal subgroups of finite index $H \trianglelefteq G$, which yields to a system of derived graphs $X_H := X(G/H, \alpha_H)$ with a cover $X_{H'} \to X_H$ for every $H' \subseteq H$.

It turns out that towers of graph covers such as the ones constructed from voltage assignments behave in a way which is very similar to towers of number fields. This can be seen for instance by analyzing the variation of the *Picard group* of graphs varying in a tower, which parallels the variation of the class group of number fields. More precisely, given a finite graph Y we denote by $\kappa(Y)$ the number of spanning trees of Y, which is also the cardinality of its Picard group. When $Y = X_H$ varies in a tower constructed from a voltage assignment $\alpha\colon \mathbf{E}_X \to G$ thanks to the procedure described above, it turns out that the evolution of $\kappa(X_H)$ is linked by an element of the group ring $\mathbb{Z}[G]$. More precisely, fix an order $V_X = \{v_1, \dots, v_g\}$, and denote by $D_X \in \mathbb{Z}^{g \times g}$ the valence matrix of X, whose (j,j)-th entry is $\# E_{X,v_j}$. Moreover, consider the matrix

$$A_\alpha := \left(\sum_{\substack{e \in E_X \\ (o(e),t(e))=(v_i,v_j)}} [\alpha(e)] \right)_{i,j=1,\dots,g} \in \mathbb{Z}[G]^{g \times g},$$

and define the *Ihara element* associated to α as $\mathcal{I}_\alpha := \det(D_X - A_\alpha) \in \mathbb{Z}[G]$. This is the sought-for element which is related to the growth rate of the number of spanning trees $\kappa(X_H)$, as H varies within the family of subgroups of finite index of G. This relation is particularly tight when G is abelian, and even more so when $G = \mathbb{Z}$, as shown in the following theorem.

Theorem 3.4 (P. & Vallières)**.** *Let X be a finite graph such that $\#\mathbf{E}_X \neq 2 \cdot \# V_X$, and let $\alpha\colon \mathbf{E}_X \to \mathbb{Z}$ be a voltage assignment. Let $\mathcal{I}_\alpha \in \mathbb{Z}[\mathbb{Z}] \cong \mathbb{Z}[t^{\pm 1}]$ be the Ihara element associated to α, and consider the integers $b := \operatorname{ord}_{t=0}(\mathcal{I}_\alpha)$ and $e := \operatorname{ord}_{t=1}(\mathcal{I}_\alpha)$. Let $J_\alpha := t^{-b}(t-1)^{-e}\mathcal{I}_\alpha$, so that $J_\alpha \in \mathbb{Z}[t]$ and $J_\alpha(0) \cdot J_\alpha(1) \neq 0$. Then*

$$\kappa(X_{N\mathbb{Z}}) = (-1)^{b(N-1)} \cdot \kappa(X) \cdot N^{e-1} \cdot \frac{\Delta_N(J_\alpha)}{\Delta_1(J_\alpha)},$$

where $\{\Delta_N(J_\alpha)\}_{N=1}^{+\infty}$ is the Pierce-Lehmer sequence defined in (1.4).

The proof of Theorem 3.4 makes crucial use of the *Ihara zeta function* Z_X associated to a finite graph X. To define it, fix an order $V_X = \{v_1, \dots, v_g\}$ and let $D_X := \operatorname{diag}(\#\mathbf{E}_{X,v_1}, \dots, \#\mathbf{E}_{X,v_g})$ be the valency matrix, and $A_X := (\#\{e\colon o(e) = v_i,\ t(e) = v_j\})_{i,j=1,\dots,g}$ be the adjacency matrix of X. This allows us to define the polynomial $h_X(u) := \det(\mathrm{Id}_g - A_X u + (D_X - \mathrm{Id}_g)u^2)$, and the *Ihara zeta function*

$$Z_X(u) := \frac{1}{(1-u^2)^{-\chi(X)} \cdot h_X(u)}, \tag{3.5}$$

where $\chi(X) := \# V_X - \#\mathbf{E}_X/2$ is the Euler characteristic of X. The polynomial $h_X(u)$ can be related to the number of spanning trees $\kappa(X)$, thanks to the formula

$$h_X'(1) = -2\chi(X)\kappa(X),$$

which is due to Hashimoto [32, Theorem B].

The definition of Ihara's zeta function admits a generalization to the equivariant setting. More precisely, let $Y \to X$ be a Galois cover of finite graphs, write $G := \operatorname{Gal}(Y/X)$ and fix a section $\tau\colon V_X \to V_Y$ of the surjection $V_Y \twoheadrightarrow V_X$. Then, given $\sigma \in G$ we let $\tilde{A}(\sigma) := (\#\{e \in \mathbf{E}_Y\colon o(e) = \tau(v_i),\ t(e) = \sigma(\tau(v_j))\})_{i,j=1,\dots,g}$. These matrices $\tilde{A}(\sigma)$ can be used to associate to every representation $\rho\colon G \to \mathrm{GL}_{d_\rho}(\mathbb{C})$ of degree d_ρ two other matrices

$$\tilde{A}_\rho := \sum_{\sigma \in G} \tilde{A}(\sigma) \otimes \rho(\sigma) \quad \text{and} \quad D_\rho := D_X \otimes \mathrm{Id}_{d_\rho}, \tag{3.6}$$

where $\otimes$ denotes the *Kronecker product* of matrices. Then, the *Artin-Ihara L-function* associated to ρ can be defined by introducing the polynomial $h_{Y/X}(u,\rho) := \det(\mathrm{Id}_{gd_\rho} - \tilde{A}_\rho \cdot u + (D_\rho - \mathrm{Id}_{gd_\rho})u^2)$, and setting

$$L_{Y/X}(u,\rho) := \frac{1}{(1-u^2)^{-\chi(X)d_\rho} \cdot h_{Y/X}(u,\rho)},$$

which closely resembles (3.5). These L-functions satisfy the *Artin formalism*, which means that

$$Z_Y(u) = Z_X(u) \cdot \prod_{\rho \in \mathrm{Irr}(G) \setminus \{\rho_0\}} L_{Y/X}(u, \rho)^{d_\rho},$$

where $\mathrm{Irr}(G)$ denotes the set of equivalence classes of irreducible representations of G, and ρ_0 is the trivial representation. Therefore we have that

$$h_Y(u) = h_X(u) \cdot \prod_{\rho \in \mathrm{Irr}(G) \setminus \{\rho_0\}} h_{Y/X}(u, \rho)^{d_\rho}, \tag{3.7}$$

because $\chi(Y) = \#G \cdot \chi(X) = (\sum_{\rho \in \mathrm{Irr}(G)} d_\rho^2)\chi(G)$. Differentiating (3.7) on both sides, and evaluating the result at $u = 1$ we see that

$$\#G \cdot \kappa(Y) = \kappa(X) \cdot \prod_{\rho \in \mathrm{Irr}(G) \setminus \{\rho_0\}} h_{Y/X}(1, \rho)^{d_\rho}, \tag{3.8}$$

because $h_X(1) = 0$. With these formulas at hand, we are finally ready to prove Theorem 3.4.

Proof. Since $G_N := \mathbb{Z}/N\mathbb{Z}$ is abelian, all its irreducible representations are given by characters. Therefore (3.8) implies that

$$N \cdot \kappa(X_{N\mathbb{Z}}) = \kappa(X) \cdot \prod_{\psi \in G_N^\vee \setminus \{\psi_0\}} h_{X_{N\mathbb{Z}}/X}(1, \psi),$$

where ψ_0 denotes the trivial character of G_N. Now, for every character $\psi \colon \mathbb{Z}/N\mathbb{Z} \to \mathbb{C}^\times$ one can see that

$$\tilde{A}_\psi := \sum_{\sigma \in \mathbb{Z}/N\mathbb{Z}} \psi(\sigma) \cdot \tilde{A}(\sigma) = \left(\sum_{\substack{e \in E_X \\ (o(e), t(e)) = (v_i, v_j)}} \psi(\alpha(e)) \right) \in \mathbb{C}^{g \times g},$$

which implies that $h_{X_{N\mathbb{Z}}/X}(1, \psi) = \det(D_X - \tilde{A}_\psi) = \mathcal{I}_\alpha(\psi(1))$. Therefore

$$N \cdot \kappa(X_{N\mathbb{Z}}) = \kappa(X) \cdot \prod_{\psi \in G_N^\vee \setminus \{\psi_0\}} \mathcal{I}_\alpha(\psi(1)) = \kappa(X) \cdot \prod_{\zeta \in W_N \setminus \{1\}} \mathcal{I}_\alpha(\zeta),$$

because when ψ varies among the non-trivial characters of $\mathbb{Z}/N\mathbb{Z}$, the value $\psi(1)$ ranges over all the possible non-trivial roots of unity whose order divides N. To conclude, notice that

$$\begin{aligned} \prod_{\zeta \in W_N \setminus \{1\}} \mathcal{I}_\alpha(\zeta) &= \prod_{\zeta \in W_N \setminus \{1\}} \zeta^b (\zeta - 1)^e J_\alpha(\zeta) = (-1)^{b(N-1)} \cdot N^e \cdot \mathrm{Res}\left(J_\alpha, \frac{t^N - 1}{t - 1} \right) \\ &= (-1)^{b(N-1)} \cdot N^e \cdot \frac{\Delta_N(J_\alpha)}{\Delta_1(J_\alpha)}, \end{aligned}$$

as follows from the definition of J_α. □

Combining Theorem 3.4 with Theorem 3.3, one can immediately describe the p-adic valuation of the number of spanning trees inside any $\mathbb{Z}$-tower of finite graphs. More precisely, fix a finite graph X such that $\chi(X) \neq 0$, a voltage assignment $\alpha \colon \mathbf{E}_X \to \mathbb{Z}$, and a prime number p. Then, we let:

- $\beta_1, \dots, \beta_K$ be the p-adic roots of J_α that lie on the unit circle of $\overline{\mathbb{Q}}_p$;
- $M_1, \dots, M_K$ be the multiplicative orders of $\pi_p(\beta_1), \dots, \pi_p(\beta_K) \in \overline{\mathbb{F}}_p^\times$;

- $\gamma_k := \tau_p(\pi_p(\beta_k))$ for every $k \in \{1, \dots, K\}$;
- $r_k := \min(\{r \in \mathbb{N} \colon p^r(p-1)\mathrm{ord}_p(\beta_k - \gamma_k) > 1\})$ for every $k \in \{1, \dots, K\}$.

Then, for every subset $\mathbf{k} \subseteq \{1, \dots, K\}$ and every $1 \leq r \leq R_p(X, \alpha) := \max(r_1, \dots, r_K)$ we define the sequence

$$\mathcal{S}_p(X, \alpha, \mathbf{k}, r) := \left\{ N \geq 1 \;\middle|\; \begin{array}{r} M_k \mid N, \ \forall k \in \mathbf{k} \\ M_{k'} \nmid N, \ \forall k' \in \{1, \dots, K\} \setminus \mathbf{k} \\ r = \min(\mathrm{ord}_p(N), R_p(X, \alpha)) \end{array} \right\},$$

so that these sequences $\mathcal{S}_p(X, \alpha, \mathbf{k}, r)$ are pairwise disjoint, and their union coincides with $\mathbb{Z}_{\geq 1}$. To conclude, we can show thanks to Theorems 3.3 and 3.4 that for every sequence $\mathcal{S}_p(X, \alpha, \mathbf{k}, r)$ there exist three invariants $\mu_p(X, \alpha), \lambda_p(X, \alpha, \mathbf{k})$ and $\nu_p(X, \alpha, \mathbf{k}, r)$ such that

$$\mathrm{ord}_p(\kappa(X_{N\mathbb{Z}})) = \mu_p(X, \alpha) \cdot N + \lambda_p(X, \alpha, \mathbf{k}) \cdot \mathrm{ord}_p(N) + \nu_p(X, \alpha, \mathbf{k}, r)$$

for every $N \in \mathcal{S}_p(X, \alpha, \mathbf{k}, r)$.

These invariants can be explicitly computed as follows. First of all, we define

$$\mu_p(X, \alpha) := -m_p(J_\alpha) / \log(p),$$

which depends only on the graph X, the voltage assignment α and the prime p. Then, for every subset $\mathbf{k} \subseteq \{1, \dots, K\}$ we define

$$\lambda_p(X, \mathfrak{m}, \alpha) := \#\mathbf{k} + \mathrm{ord}_{t=1}(\mathcal{I}_\alpha) - 1$$

and finally for every $r \in \{1, \dots, R_p(X, \alpha)\}$ we define

$$\nu_p(X, \alpha, \mathbf{k}, r) := c_p(X, \alpha) + \begin{cases} \sum_{k \in \mathbf{k}} \mathrm{ord}_p(\beta_k - \gamma_k), \text{ if } p \nmid 2 \cdot \mathrm{Disc}(J_\alpha) \text{ or } r = 0 \\ \sum_{k \in \mathbf{k}} \mathrm{ord}_2(\beta_k^2 - \gamma_k^2), \text{ if } p = 2, \ r \geq 1 \text{ and } 2 \nmid \mathrm{Disc}(J_\alpha) \\ \sum_{k \in \mathbf{k}} \mathrm{ord}_p(\beta_k^{p^{\min(r, r_k)}} - \gamma_k^{p^{\min(r, r_k)}}), \text{ if } p \mid \mathrm{Disc}(J_\alpha) \end{cases}$$

where $c_p(X, \alpha) := \mathrm{ord}_p(\kappa(X)) - \mathrm{ord}_p(\Delta_1(J_\alpha))$.

In particular, for every prime number ℓ we see that there exists some $n_0 \geq 1$ such that the sequence of powers $\{\ell^n\}_{n=n_0}^{+\infty}$ is contained inside a unique sequence of the form $\mathcal{S}_p(X, \alpha, \mathbf{k}, r)$. Therefore, we see that in the ℓ-adic tower

$$X_{\ell^\infty} \colon \ \cdots \to X_{\ell^n} \to X_{\ell^{n-1}} \to \ldots \to X_\ell \to X_1 = X$$

the p-adic valuations of the number of spanning trees are governed by invariants depending only on ℓ and p, as it happens in the Iwasawa theory of number fields.

Homology growth and Pierce-Lehmer sequences

Let us conclude this lecture by surveying the relations between Mahler measures and torsion homology growth. In general, if M is a connected, compact topological manifold, one can wonder about the evolution of the finite torsion homology groups $H_1(M_H; \mathbb{Z})_{\mathrm{tors}}$ as M_H varies amongst the finite covers of M, corresponding to normal subgroups of finite index $H \subseteq \pi_1(M)$. Something particularly interesting happens when M is the *exterior* of an *oriented link* $\ell = \ell_1 \cup \cdots \cup \ell_n$ inside the three sphere S^3. More precisely,

$\ell_1, \ldots, \ell_n \subseteq S^3$ are *knots*, *i.e.* embeddings of S^1, which do not intersect, and $M = S^3 \setminus N(\ell)^\circ$, for some *regular neighbourhood* $N(\ell) \subseteq S^3$ of ℓ. If $N(\ell)$ is small enough, then $H_1(M;\mathbb{Z}) \cong \mathbb{Z}^n$, and we obtain a corresponding abelian universal cover $\pi_\infty \colon M_\infty \twoheadrightarrow M_1 := M$, with group of deck transformations isomorphic to $\mathbb{Z}^n$. Therefore, for every base point $b \in M_1$, the first homology group $A(\ell) := H_1(M_\infty \setminus \pi_\infty^{-1}(b);\mathbb{Z})$ is a module over the group ring $\mathbb{Z}[H_1(M;\mathbb{Z})] \cong \mathfrak{R}_n$ where $\mathfrak{R}_n := \mathbb{Z}[x_1^{\pm 1}, \ldots, x_n^{\pm 1}]$. This module turns out to be *finitely presented*, which means that there exist two integers $a, b \geq 1$ and an exact sequence of $\mathfrak{R}_n$-modules $\mathfrak{R}_n^a \to \mathfrak{R}_n^b \to A(\ell) \to 0$, to which we can naturally associate a *presentation matrix* in $\mathfrak{R}_n^{a \times b}$. One can see that for every integer $k \geq 1$, any finitely presented $\mathfrak{R}_n$-module admits a presentation matrix with $b > k$ columns, and $a \geq b - k$ rows. Then, for every $k \geq 1$ one defines the k-th *Alexander polynomial* $\Delta_\ell^{(k)} \in \mathfrak{R}_n$ as the greatest common divisor of the elements of the ideal generated by the minors of size $b - (k+1)$ of any presentation matrix for $A(\ell)$ with $b > k+1$ columns and $a \geq b - (k+1)$ rows. It turns out that each polynomial $\Delta_\ell^{(k)}$ is well defined up to the multiplication by a unit of $\mathfrak{R}_d$, which is forcefully a monomial $\varepsilon \cdot \underline{x}_n^v$ for some $\varepsilon \in \{\pm 1\}$ and $v \in \mathbb{Z}^n$. In particular, each of these polynomials has a well defined Mahler measure. The following result, due to Silver and Williams [65], shows how the Mahler measure of the *Alexander polynomial* $\Delta_\ell := \Delta_\ell^{(0)}$ can be used to detect the growth rate of torsion in the first homology groups of the manifolds canonically associated to the subgroups of finite index of $\mathbb{Z}^n$.

Theorem 3.9 (Silver & Williams). *Let $\ell \subseteq S^3$ be an oriented link with n components, and let M be the exterior of ℓ. Moreover, for every subgroup $\Lambda \subseteq \mathbb{Z}^n$ let $\rho(\Lambda) := \min\{\|v\|_\infty \colon v \in \Lambda \setminus \{0\}\}$. Then, we have that*

$$\limsup_{\rho(\Lambda) \to +\infty} \frac{1}{|\mathbb{Z}^n/\Lambda|} \log(\# H_1(M_\Lambda;\mathbb{Z})_{tors}) = m(\Delta_\ell), \tag{3.10}$$

where Λ ranges over the subgroups of finite index of $\mathbb{Z}^n$, while M_Λ is the $\mathbb{Z}^n/\Lambda$-cover of M associated to Λ, and Δ_ℓ denotes the Alexander polynomial associated to ℓ.

Proof. The main idea behind Silver and Williams's proof is to relate the numbers $\# H_1(M_\Lambda;\mathbb{Z})_{\text{tors}}$ to the numbers of Λ-periodic points of a certain $\mathbb{Z}^n$-action on a compact abelian group. The latter is then seen to have the same entropy as the dynamical system associated to the Alexander module $A(\ell)$, which coincides with the Mahler measure of Δ_ℓ. To conclude, Silver and Williams use a result relating the growth of preperiodic points and entropy, which is due to Lind, Schmidt and Ward (see [62, Theorem 21.1]). More precisely, if X is a compact abelian group supporting a $\mathbb{Z}^n$-action $\alpha \colon \mathbb{Z}^n \to \operatorname{Aut}(X)$ which satisfies the descending chain condition and has finite entropy $h(X)$, we have that

$$\limsup_{\rho(\Lambda) \to +\infty} \frac{1}{|\mathbb{Z}^n/\Lambda|} \log(\#(\operatorname{Per}_\Lambda(X)/\operatorname{Per}_\Lambda(X)^0)) = h(X), \tag{3.11}$$

where $\operatorname{Per}_\Lambda(X) := \{x \in X \colon \alpha_v(x) = x, \ \forall v \in \Lambda\}$, and $\operatorname{Per}_\Lambda(X)^0$ denotes the connected component of the identity. $\square$

Remark 3.12. Given a subgroup $\Lambda \subseteq \mathbb{Z}^n$, let $\mathbb{T}^n[\Lambda] := \{\xi \in \mathbb{T}^n \colon \xi^v = 1, \ \forall v \in \mathbb{Z}^n\}$. In particular, if Λ has finite index in $\mathbb{Z}^n$, then $\mathbb{T}^n[\Lambda]$ is a finite group of roots of unity. Now, for every Laurent polynomial $P \in \mathbb{C}[x_1^{\pm 1}, \ldots, x_n^{\pm 1}] \setminus \{0\}$, one has that

$$\limsup_{\rho(\Lambda) \to +\infty} \frac{1}{|\mathbb{Z}^n/\Lambda|} \sum_{\substack{\xi \in \mathbb{T}^n[\Lambda] \\ P(\xi) \neq 0}} \log(P(\xi)) = m(P), \tag{3.13}$$

if Λ ranges over subgroups of finite index. This can be seen as a sort of generalization of Theorem 1.6, and is intimately related to (3.10) and (3.11). In particular, the limit *supremum* appearing in (3.13) is conjectured to be an actual limit, which would entail that the growth of periodic points (3.11) and of homology (3.10) can also be measured by actual limiting statements. To this day, converting (3.13) to an actual limit statement seems out of reach, as the only available strategy to prove limits of this form uses deep results in Diophantine approximation, which are unavailable in higher dimension. Nevertheless, Dimitrov [21] has shown that

$$\lim_{N \to +\infty} \frac{1}{N^n} \sum_{\substack{\xi \in \mathbb{T}^n[N\mathbb{Z}^n] \\ P(\xi) \neq 0}} \log(P(\xi)) = m(P),$$

which allows one to replace the limit supremum in (3.13) with an actual limit if one considers only the subgroups $\Lambda \in \{N\mathbb{Z}^n \colon N \geq 1\}$. Moreover, Lind, Schmidt and Verbitskiy [47] have proved that this can be done for any sequence of subgroups if the polynomial P is *essentially atoral*, which means that the Zariski closure of the set of toric points $V_P(\mathbb{C}) \cap \mathbb{T}^n := \{\xi \in \mathbb{T}^n \colon P(\xi) = 0\}$ is a union of components having codimesion at least two and of translates of subgroups of $\mathbb{G}_m^n$ by torsion points. Finally, one can also consider Galois orbits of torsion points, for which a similar statement, with a control on the rate of convergence, has been proven by Dimitrov and Habegger [23].

The work of Silver and Williams resumed in Theorem 3.9 allows one to reduce Lehmer's problem, outlined in Question 1.8, to a purely topological question, as shown successively by the same authors. More precisely, if ℓ is an oriented link inside an oriented and closed manifold of dimension three, then any sufficiently small regular neighbourhood $N(\ell)$ is homeomorphic to $\ell \times \mathbb{D}$, where $\mathbb{D} \subseteq \mathbb{C}$ denotes the unit disk, and one says that ℓ is *fibred* if the projection $\partial N(\ell) \cong \ell \times \mathbb{T} \to \mathbb{T}$ extends to a locally trivial fibration of the exterior of ℓ. Then, work of Kanenobu shows that for every polynomial $P \in \mathbb{Z}[t]$ which is reciprocal and has even degree, there exists a fibred link ℓ with only two components such that P coincides with $\Delta_\ell(t,t)$ up to a monomial of the form $\varepsilon \cdot t^k$, where $\varepsilon \in \{\pm 1\}$ and $k \in \mathbb{Z}$. Since one of the two components of the link ℓ can be taken to be unknotted, Silver and Williams show that Question 1.8 has a positive answer if and only if for every fibred knot κ inside a *lens space* $L(n,1)$, the cardinalities of the torsion subgroups of the first homology groups of the canonical $\mathbb{Z}$-tower of three-dimensional manifolds constructed over the exterior of κ are either periodic or have a strictly positive growth rate. Let us recall in particular that the *lens space* $L(n,1)$ is defined as the quotient of the unit sphere in $\mathbb{C}^2$ by the action of the group of n-th roots of unity $\mu_n \cong \mathbb{Z}/n\mathbb{Z}$ given by $\zeta * (z_1, z_2) = (\zeta \cdot z_1, z_2)$. To conclude, let us mention that more recent work of Silver and Williams [67] provided a graph-theoretic question which is also equivalent to Lehmer's problem. To do so, they consider *signed graphs*, in which each edge is endowed with a sign, and define analogues of the Ihara elements $\mathcal{I}_\alpha$ for $\mathbb{Z}$-valued voltage assignments on these signed graphs. Then, they show a graph-theoretic analogue of Kanenobu's result, which guarantees that every monic and reciprocal polynomial can be obtained as one of these signed Ihara elements.

4 Mahler measures and special values of L-functions

In this last lecture, we are going to see how Mahler measures can be related to special values of L-functions of arithmetic objects.

L-functions

To see what these are, recall that to every smooth and projective variety X defined over the rationals, every integer $0 \leq i \leq 2\dim(X)$ and every rational prime $\ell \in \mathbb{N}$ one can associate an *étale cohomology group* $H^i_\ell(X) := H^i_{\text{ét}}(X_{\overline{\mathbb{Q}}}; \mathbb{Q}_\ell)$, on which the absolute Galois group $G_{\mathbb{Q}} := \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ acts naturally. It turns out that this Galois representation $\rho\colon G_{\mathbb{Q}} \to \mathrm{Aut}(H^i_\ell(X))$ can be recovered, at least up to semi-simplification, by considering the images $\rho(\mathrm{Frob}_p)$ of the *geometric Frobenius* conjugacy classes associated to those rational primes p at which ρ is *unramified*, which means that $\rho(I_p) = \{1\}$, where I_p denotes the *inertia group* at p. In particular, Frob_p is an automorphism of $\overline{\mathbb{Q}}$ that lifts, up to elements of I_p, the *inverse* of the automorphism $\overline{\mathbb{F}}_p \to \overline{\mathbb{F}}_p$ defined by $x \mapsto x^p$. Now, using the aforementioned Frobenius conjugacy classes, one can define the L-function $L(H^i_\ell(X), s)$ as a formal Euler product

$$L(H^i_\ell(X), s) := \prod_p \frac{1}{\det(1 - \mathrm{Frob}_p \cdot p^{-s} \mid \underline{D}_p(H^i_\ell(X)))}, \tag{4.1}$$

where $\underline{D}_p(H^i_\ell(X)) = H^i_\ell(X)^{I_p}$ if $\ell \neq p$, while $\underline{D}_\ell(H^i_\ell(X)) := (B_{\mathrm{cris},\ell} \otimes H^i_\ell(X))^{\mathrm{Gal}(\overline{\mathbb{Q}}_\ell/\mathbb{Q}_\ell)}$ is defined in terms of the ring of *crystalline ℓ-adic periods* $B_{\mathrm{cris},\ell}$, as explained in [28]. Notice that all the characteristic polynomials

$$\det(1 - \mathrm{Frob}_p \cdot t \mid \underline{D}_p(V)) \in \mathbb{Q}_\ell[t]$$

appearing in (4.1) are conjectured to have rational coefficients, so that $L(H^i_\ell(X), s)$ would as well define an Euler product over the rational and complex numbers. Assuming this conjecture, Deninger [18] observes that one can also define another formal Euler product

$$L(H^i(X), s) := \prod_p \frac{1}{\det(1 - \mathrm{Frob}_p \cdot p^{-s} \mid \underline{D}_p(H^i_p(X)))},$$

which does not depend on the choice of any auxiliary prime, and is conjectured to coincide with each of the L-functions $L(H^i_\ell(X), s)$. One also conjectures that there exists some $\sigma_0 \in \mathbb{R}$ such that the formal Euler product defining $L(H^i_\ell(X), s)$ actually converges for $\Re(s) > \sigma_0$. The resulting holomorphic function is expected to have a meromorphic continuation to the whole complex plane $\mathbb{C}$, which should satisfy a functional equation relating the function $L(H^i(X), s)$ to the function $L(H^i(X), i + 1 - s)$ by means of a product of factors obtained from Euler's gamma function $\Gamma(s)$.

Example 4.2. When $X = \mathrm{Spec}(\mathbb{Q})$ is just a point, we have that $L(H^0(X), s) = \zeta(s)$, where ζ denotes Riemann's zeta function. Similarly, if $X = \mathrm{Spec}(K)$, where K is a number field, then $L(H^0(X), s) = \zeta_K(s)$ coincides with the Dedekind zeta function associated to K. If $K = \mathbb{Q}(\sqrt{d})$ is a quadratic field, we have the factorization $\zeta_K(s) = \zeta(s) \cdot L(\chi_d, s)$, where $L(\chi_d, s)$ is the L-function associated to the quadratic character $n \mapsto \left(\frac{d}{n}\right)$. In particular, for $\Re(s) > 1$ one can define the L-function $L(\chi_d, s)$ by the convergent Dirichlet series $L(\chi_d, s) = \sum_{n=1}^{+\infty} \chi_d(n) \cdot n^{-s}$. Finally, if E is an elliptic curve defined over $\mathbb{Q}$, the L-function $L(E, s) := L(H^1(E), s)$ can be defined for $\Re(s) > \frac{3}{2}$ by the convergent Euler product

$$L(E, s) = \prod_{p \nmid \mathfrak{f}_E} \frac{1}{1 - a_p(E) p^{-s} + p^{1-2s}} \cdot \prod_{p \in \mathcal{B}_{sm}(E)} \frac{1}{1 - p^{-s}} \cdot \prod_{p \in \mathcal{B}_{nsm}(E)} \frac{1}{1 + p^{-s}}$$

where $\mathcal{B}_{sm}(E)$ denotes the set of primes of bad, split multiplicative reduction for E, while $\mathcal{B}_{nsm}(E)$ denotes the set of primes of bad, non-split multiplicative reduction of E, and $\mathfrak{f}_E$ denotes the conductor of E. Moreover, for every prime $p \nmid \mathfrak{f}_E$, we define $a_p(E) := p + 1 - \#\tilde{E}(\mathbb{F}_p)$, where $\tilde{E}$ denotes the reduction of E modulo p.

Special values of L-functions

Given a smooth and projective variety X defined over the rationals, it turns out that each of the L-functions $L(H^i(X), s)$ can be reconstructed from its *special values* at the integers, as was shown by Deninger [20]. More generally, given a complex number $s_0 \in \mathbb{C}$ and a meromorphic function $f \colon \mathbb{C} \dashrightarrow \mathbb{C}$, the *special value* of f at $s = s_0$ is defined as

$$f^*(s_0) := \lim_{s \to s_0} \frac{f(s)}{(s - s_0)^{\operatorname{ord}_{s=s_0}(f)}} \in \mathbb{C}^\times,$$

where $\operatorname{ord}_{s=s_0}(f) \in \mathbb{Z}$ is the order of vanishing, or the order of pole, or the function f at $s = s_0$. In particular, the special value $f^*(s_0)$ is the first non-vanishing coefficient in the Laurent series expansion of f around $s = s_0$. It is perhaps even more surprising that not only one can reconstruct $L(H^i(X), s)$ from its special values, but that the latter turn out to be related to the arithmetic of X in a quite intricate way.

The first instance of such a relation is given by the *analytic class number formula*, which says that for every number field K one can compute the special value of ζ_K at $s = 0$ as

$$\zeta_K^*(0) = -\frac{\#\mathrm{Cl}(\mathcal{O}_K)}{\#(\mathcal{O}_K^\times)_{\mathrm{tors}}} \cdot \mathrm{Reg}_K, \tag{4.3}$$

where $\mathrm{Cl}(\mathcal{O}_K)$ denotes the *class group* of the ring of integers $\mathcal{O}_K$, while $\mathcal{O}_K^\times$ denotes the group of units of $\mathcal{O}_K$, and Reg_K is the *regulator* of K, which is defined as the determinant of a matrix of the form $(\log|\sigma_i(\alpha_j)|)_{i,j=1,\dots,r}$. Here $r = r_1 + r_2 - 1$, where r_1 denotes the number of real embeddings of K, while r_2 denotes the number of pairs of complex conjugate embeddings of K, so that $[K\colon \mathbb{Q}] = r_1 + 2r_2$. Moreover, $\alpha_1, \dots, \alpha_r$ are a basis of $\mathcal{O}_K^\times / (\mathcal{O}_K^\times)_{\mathrm{tors}}$, while $\sigma_1, \dots, \sigma_{r+1}$ are embeddings of K inside $\mathbb{C}$ which represent all the conjugacy classes of embeddings $K \hookrightarrow \mathbb{C}$. Note in particular that to define the regulator we only need r embeddings, and the definition does not depend on which embedding we choose to leave out.

The aforementioned analytic class number formula (4.3) admits a plethora of generalizations, most of which are conjectural. The most famous of them is probably the *Birch and Swinnerton-Dyer conjecture*, which relates the central critical special value $L^*(E, 1)$ associated to the L-function of an elliptic curve E (or, more generally, of an abelian variety A) to various quantities related to E, such as a regulator, a product of factors coming from the primes of bad reduction, and the cardinality of the torsion subgroup of the group of rational points $E(\mathbb{Q})$. Despite the great interest behind this conjecture, we will not mention it further, since we will concentrate on special values lying *outside* the critical strip. We refer the interested reader to Tate's original article [70] for a precise formulation of the conjecture, and to Coates's lecture notes [16] for a more modern treatment.

When we look at special values outside the critical strip, Beilinson [2] and Bloch and Kato [3, 29] have proposed far-reaching generalizations of the class number formula, and the Birch and Swinnerton-Dyer conjecture. More precisely, fix two integers $i \geq 1$ and $n > \frac{i+1}{2}$, and a smooth and projective variety X defined over the rationals, such that $i < 2\dim(X)$. In order to study the special value $L^*(H^{i-1}(X), n)$, Beilinson introduced a *regulator map*

$$\mathrm{reg}_\infty \colon H^i_{\mathcal{M}}(X; \mathbb{Q}(n)) \to H^i_{\mathcal{D}}(X_{\mathbb{R}}; \mathbb{R}(n)) \tag{4.4}$$

between two very complicated cohomology groups. More precisely, the source of Beilinson's regulator is the *motivic cohomology* group $H^i_{\mathcal{M}}(X, \mathbb{Q}(n))$, which can be defined in several different ways. When U is a smooth variety, each class $[Z] \in H^i_{\mathcal{M}}(U, \mathbb{Q}(n))$ can for example be represented as a cycle of codimension n inside the variety $U \times \square^{2n-i}$, where $\square := \mathbb{P}^1 \setminus \{1\}$, which meets transversely the *faces* $U \times \partial(\square^n)$, where $\partial(\square^n) \subseteq \square^n$ is the union of all the hyperplanes cut out by the equations $t_i = 0$ or $t_i = \infty$, for $i \in \{1, \dots, n\}$. On the other hand, the target of Beilinson's regulator map is *Deligne cohomology*, which is a cohomology theory that can be regarded as a twisted version of de Rham cohomology. If U is a smooth, non necessarily projective variety, a class $[\alpha] \in H^i_{\mathcal{D}}(U_{\mathbb{R}}; \mathbb{R}(n))$ can be represented by a smooth differential form α defined on U, which has *logarithmic singularities* along the boundary, meaning that there exists a smooth, projective compactification $X \hookleftarrow U$ such that, locally around every point of $D := X \setminus U$, the differential form α can be expressed in terms of regular differential forms on X, the functions $\log|z_j|$ and the forms $\frac{dz_j}{z_j}$ and $\frac{d\overline{z}_j}{\overline{z}_j}$, where $z_1 \cdots z_r = 0$ is a local equation for D. The degree of the differential form α equals $i-1$ whenever $i \leq 2n-1$, and equals i otherwise. Moreover, α can be written as a sum of forms of type (p,q) with $p, q < n$ whenever $i \leq 2n-1$, and as a sum of forms of type (p,q) with $p, q \geq n$ otherwise. Finally, $d(\alpha) = 0$ whenever $i \geq 2n$, whereas $dd^c(\alpha) = 0$ if $i = 2n-1$, and only $\mathrm{pr}_n(d\alpha) = 0$ otherwise, where pr_n denotes the projection which discards all the components of type (p,q) such that either $p \geq n$ or $q \geq n$.

Going back to Beilinson's work [2], he predicted the following relation between the special value at $s = n$ of the L-function $L(H^{i-1}(X), s)$ and the regulator map portrayed in (4.4).

Conjecture 4.5 (Beilinson)**.** *Let X be a smooth and projective variety, and fix two integers $i, n \in \mathbb{N}$ such that $0 \leq i - 1 \leq 2\dim(X)$ and $n > \frac{i}{2} - 1$. Then, if $H^i_{\mathcal{M},\mathbb{Z}}(X; \mathbb{Q}(n)) \subseteq H^i_{\mathcal{M}}(X; \mathbb{Q}(n))$ denotes the subspace consisting of those classes $[Z] \in H^i_{\mathcal{M}}(X; \mathbb{Q}(n))$ that have "good reduction everywhere", the regulator map reg_∞ should induce an isomorphism $H^i_{\mathcal{M},\mathbb{Z}}(X; \mathbb{Q}(n)) \otimes_{\mathbb{Q}} \mathbb{R} \xrightarrow{\sim} H^i_{\mathcal{D}}(X_{\mathbb{R}}; \mathbb{R}(n))$ whose determinant, with respect to the canonical $\mathbb{Q}$-structures on the source and on the target, coincides with the class of $L^*(H^{i-1}(X), n)$ in $\mathbb{R}^\times/\mathbb{Q}^\times$.*

Remark 4.6. The notion of having *good reduction everywhere* for a class $[Z] \in H^i_{\mathcal{M}}(X; \mathbb{Q}(n))$, needs to be properly defined. One can do so either by fixing a sufficiently regular model of X over the integers, or by using some Galois cohomology, similarly to what one does to define the Tate-Shafarevich group of an elliptic curve. We refer the interested reader to the works of Scholl [64] and Scholbach [63], where these notions have been properly defined.

Remark 4.7. The $\mathbb{Q}$-structure on the Deligne cohomology group $H^i_{\mathcal{D}}(X_{\mathbb{R}}; \mathbb{R}(n))$ can actually be chosen in two different ways, corresponding to the two different special values $L^*(H^{i-1}(X), n)$ and $L^*(H^{i-1}, i-n)$, which are conjecturally related by the functional equation satisfied by $L(H^{i-1}(X), s)$.

Remark 4.8. Conjecture 4.5 determines the special value $L^*(H^{i-1}(X), n)$ only up to a rational number. To fix this, Bloch and Kato [3, 29] have shown how one can associate to each rational prime p a *p-adic regulator map* reg_p, whose determinant can be used to pin down the p-adic valuation of the rational factor which is left undetermined by Conjecture 4.5. Another approach, proposed by Kahn [37, Page 394], would consist in looking at integral structures on the source and the target of Beilinson's regulator map, and in taking the determinant with respect to those.

Mahler measures and special values of L-functions

We are finally ready to talk about the relations between Mahler measures and special values of L-functions. The first instance of these relations consists in the computation of the Mahler measure of $x_1 + x_2 + 1$, shown in the following proposition, which is due to Smyth [69].

Proposition 4.9 (Smyth)**.** $m(x_1 + x_2 + 1) = L'(\chi_{-3}, -1)$.

Proof. Using Jensen's formula, we see that $m(x_1+x_2+1) = \frac{1}{(2\pi i)}\int_{\mathbb{T}^1}\log\max(|x_1+1|,1)\frac{dx_1}{x_1}$. Therefore

$$\begin{aligned} m(x_1+x_2+1) &= \frac{1}{(2\pi i)}\int_{\mathbb{T}^1}\log\max(|x_1+1|,1)\frac{dx_1}{x_1} = \int_{-1/3}^{1/3}\log|e^{2\pi i t}+1|dt \\ &= \sum_{n=1}^{+\infty}\frac{(-1)^{n-1}}{n}\Re\left(\int_{-1/3}^{1/3}e^{2\pi i n t}dt\right) = \frac{1}{\pi}\sum_{n=1}^{+\infty}\frac{(-1)^{n-1}\sin(2\pi n/3)}{n^2} \\ &= \frac{\sqrt{3}}{2\pi}\sum_{n=1}^{+\infty}\frac{(-1)^{n-1}\chi_{-3}(n)}{n^2} = \frac{3\sqrt{3}}{4\pi}L(\chi_{-3},2) = L'(\chi_{-3},-1), \end{aligned}$$

as we wanted to show. □

The elementary computation provided in Proposition 4.9 can be generalized to compute that

$$m(x_1+x_2+x_3+1) = -14\cdot\zeta'(-2), \tag{4.10}$$

where $\zeta(s)$ denotes Riemann's zeta function. Intrigued by these results, Boyd [8] started a long numerical search for new links between Mahler measures and special values of L-functions. This culminated with several amusing conjectural relations concerning families of Laurent polynomials, such as

$$m\left(x_1+\frac{1}{x_1}+x_2+\frac{1}{x_2}+k\right) \stackrel{?}{=} \alpha_k\cdot L^*(E_k,0), \tag{4.11}$$

where $E_k\colon y^2+kxy = x^3-2x^2+x$ is an elliptic curve which is birationally equivalent to the variety V_{P_k} cut out by the Laurent polynomial $P_k := x_1+\frac{1}{x_1}+x_2+\frac{1}{x_2}+k$ inside $\mathbb{G}_m^2$, whereas α_k is conjectured to be a non-zero rational number of relatively small height. In fact, it seems as well that α_k^{-1} should be an integer for all but finitely many values of k. Finally, let us note that this relation is conjectured to hold for every integer $k\in\mathbb{Z}\setminus\{-4,0,4\}$, but fails to be true when $k\in\{-4,0,4\}$. More precisely, Ray [55] has proven that

$$m\left(x_1+\frac{1}{x_1}+x_2+\frac{1}{x_2}+4\right) = 2\cdot L^*(\chi_{-4},-1),$$

where χ_{-4} denotes the Dirichlet character associated to the Gaussian integers. Finally, Boyd [8] has found many more families of polynomials $Q_k\in\mathbb{Z}[x_1,x_2]$ whose Mahler measure seems to be rationally proportional to $L^*(E_k',0)$, where E_k' is an elliptic curve birational to the zero locus of Q_k in $\mathbb{G}_m^2$. These families Q_k were also studied more systematically in subsequent work of Rodriguez-Villegas [57]. Moreover, Boyd [8] also found some relations between the Mahler measures of polynomials $P\in\mathbb{Z}[x_1,x_2]$ defining a curve of genus at least two and the L-function of a factor of its Jacobian, and also some more spurious relations, such as

$$m(x_2^2+k(x_1+1)x_2+x_1^3) \stackrel{?}{=} \beta_k\cdot L^*(E_k'',0)+\frac{1}{3}\log|k|, \tag{4.12}$$

where $k\in\mathbb{Z}\setminus\{0\}$ and $E_k''\colon y^2+kxy-ky = x^3$ is an elliptic curve, while β_k should again be some rational number of relatively small height.

Deninger's method

It is natural to wonder why the relations found by Smyth [69] and Boyd [8] should actually be true. To do so, Deninger [19] observed that Mahler measures can be related to a specific form of regulator integral, using once again Jensen's formula. More precisely, fix $P \in \mathbb{C}[x_1, \dots, x_n] \setminus \{0\}$. Then, we can write

$$m(P) = \frac{1}{(2\pi i)^{n-1}} \int_{\mathbb{T}^{n-1}} \left(\frac{1}{2\pi i} \int_{\mathbb{T}} \log|P(x_1, \dots, x_{n-1}, x_n)| \frac{dx_n}{x_n} \right) \frac{dx_1}{x_1} \cdots \frac{dx_{n-1}}{x_{n-1}}$$
$$= \frac{1}{(2\pi i)^{n-1}} \int_{\mathbb{T}^{n-1}} \left(\log|\tilde{P}(x_1, \dots, x_{n-1})| - \sum_{j=1}^{d} \log \min(1, |\alpha_j(x_1, \dots, x_{n-1})|) \right) \frac{dx_1}{x_1} \cdots \frac{dx_{n-1}}{x_{n-1}},$$

where we write $P = \tilde{P}(x_1, \dots, x_{n-1}) \cdot (x_n - \alpha_1(x_1, \dots, x_{n-1})) \cdots (x_n - \alpha_d(x_1, \dots, x_{n-1}))$, and we suppose that $\tilde{P} \neq 0$, which we can do thanks to an argument similar to the one appearing in the proof of Proposition 2.1. Therefore, if we define the *Deninger set*

$$\gamma_P := \{(x_1, \dots, x_n) \in (\mathbb{C}^\times)^n \colon |x_1| = \cdots = |x_{n-1}| = 1, |x_n| \leq 1\} \cap V_P(\mathbb{C}), \tag{4.13}$$

we see that

$$m(P) = m(\tilde{P}) - \frac{1}{(2\pi i)^{n-1}} \int_{\gamma_P} \log|x_n| \cdot \frac{dx_1}{x_1} \cdots \frac{dx_{n-1}}{x_{n-1}}, \tag{4.14}$$

which allows us to write $m(P) - m(\tilde{P})$ as an integral of a certain differential form over the set γ_P. More precisely, in order for this integral to be well defined, one has either to assume that γ_P is a manifold with boundary, or to use the fact that γ_P is a semi-algebraic set, and therefore can be triangulated, as proven by Hironaka [34] (see also [35, Section 2.6]). However, the differential form appearing in (4.14) is very far from being closed. To amend this, one can observe that, for $(x_1, \dots, x_n) \in \gamma_P$ we have that

$$\log|x_n| \frac{dx_1}{x_1} \cdots \frac{dx_{n-1}}{x_{n-1}} = \log|x_n| \cdot \frac{(-1)^{n-1}}{(n-1)!} \sum_{\substack{\sigma \in \mathfrak{S}_n \\ \sigma(1) = n}} \operatorname{sgn}(\sigma) \frac{dx_{\sigma(2)}}{x_{\sigma(2)}} \cdots \frac{dx_{\sigma(n)}}{x_{\sigma(n)}}$$
$$= (-1)^{n-1} \log|x_n| \cdot \sum_{j=1}^{n} \frac{(-1)^j}{n!} \sum_{\substack{\sigma \in \mathfrak{S}_n \\ \sigma(1) = n}} \operatorname{sgn}(\sigma) \frac{d\overline{x}_{\sigma(2)}}{\overline{x}_{\sigma(2)}} \cdots \frac{d\overline{x}_{\sigma(j)}}{\overline{x}_{\sigma(j)}} \cdot \frac{dx_{\sigma(j+1)}}{x_{\sigma(j+1)}} \cdots \frac{dx_{\sigma(n)}}{x_{\sigma(n)}}$$
$$= (-1)^{n-1} \sum_{j=1}^{n} \frac{(-1)^j}{n!} \sum_{\sigma \in \mathfrak{S}_n} \operatorname{sgn}(\sigma) \log|x_{\sigma(1)}| \frac{d\overline{x}_{\sigma(2)}}{\overline{x}_{\sigma(2)}} \cdots \frac{d\overline{x}_{\sigma(j)}}{\overline{x}_{\sigma(j)}} \cdot \frac{dx_{\sigma(j+1)}}{x_{\sigma(j+1)}} \cdots \frac{dx_{\sigma(n)}}{x_{\sigma(n)}}$$

where $\mathfrak{S}_n$ denotes the group of permutations on n letters. More precisely, the first equality follows from the fact that

$$\frac{dx_1}{x_1} \cdots \frac{dx_{n-1}}{x_{n-1}} = (-1)^{n-1} \operatorname{sgn}(\sigma) \frac{dx_{\sigma(2)}}{x_{\sigma(2)}} \cdots \frac{dx_{\sigma(n)}}{x_{\sigma(n)}},$$

for every permutation $\sigma \in \mathfrak{S}_n$ such that $\sigma(1) = n$, thanks to the alternating properties of the wedge product. On the other hand, the second equality follows from the fact that

$$\left. \frac{d\overline{z}}{\overline{z}} \right|_{\mathbb{T}} = \frac{d(z^{-1})}{z^{-1}} = -\frac{dz}{z},$$

while the third one follows from the fact that for every $(x_1, \dots, x_n) \in \gamma_P$ we have that $|x_j| = 1$ whenever $j \in \{1, \dots, n-1\}$. Therefore, we see that

$$m(P) - m(\tilde{P}) = \frac{(-1)^n}{(2\pi i)^{n-1}} \int_{\gamma_P} \eta_n, \tag{4.15}$$

where η_n is the $n-1$ form defined by

$$\eta_n := \sum_{j=1}^{n} \frac{(-1)^j}{n!} \sum_{\sigma \in \mathfrak{S}_n} \operatorname{sgn}(\sigma) \log|x_{\sigma(1)}| \frac{d\overline{x}_{\sigma(2)}}{\overline{x}_{\sigma(2)}} \cdots \frac{d\overline{x}_{\sigma(j)}}{\overline{x}_{\sigma(j)}} \cdot \frac{dx_{\sigma(j+1)}}{x_{\sigma(j+1)}} \cdots \frac{dx_{\sigma(n)}}{x_{\sigma(n)}},$$

which can be easily seen to have logarithmic singularities along the boundary of $\mathbb{G}_m^n$. Since

$$d(\eta_n) = \Re_n\left(\frac{dx_1}{x_1}\cdots\frac{dx_n}{x_n}\right),$$

where $\Re_n = \Re(z)$ if $2 \nmid n$ while $\Re_n(z) = \Im(z)$ if $2 \mid n$, we see that η_n defines an element $[\eta_n] \in H^n_{\mathcal{D}}(\mathbb{G}_m^n; \mathbb{R}(n))$.

Therefore, we see from (4.15) that Mahler measures can be related to regulator integrals. More precisely, suppose that:

- the set γ_P defined in (4.13) does not touch the singular points of V_P, *i.e.* $\gamma_P \subseteq V_P^{\text{reg}}(\mathbb{C})$;
- the set γ_P defined in (4.13) is closed as a topological chain, *i.e.* $\partial\gamma_P = \varnothing$;
- there exists a smooth, projective variety X and a class $\alpha \in H^n_{\mathcal{M},\mathbb{Z}}(X;\mathbb{Q}(n))$ such that $V_P^{\text{reg}} \subseteq X$ and $\{x_1,\dots,x_n\}\big|_{V_P^{\text{reg}}} = \alpha\big|_{V_P^{\text{reg}}}$.

Then (4.15) shows that $m(P) - m(\tilde{P})$ coincides with one of the entries appearing in the matrix which represents Beilinson's regulator $\text{reg}_\infty \colon H^n_{\mathcal{M},\mathbb{Z}}(X;\mathbb{Q}(n)) \otimes_{\mathbb{Q}} \mathbb{R} \to H^n_{\mathcal{D}}(X_{\mathbb{R}};\mathbb{R}(n))$. In particular, if we have that

$$\dim_{\mathbb{R}}(H^n_{\mathcal{D}}(X_{\mathbb{R}};\mathbb{R}(n))) = 1,$$

then, according to Beilinson's conjecture, the difference of Mahler measures $m(P) - m(\tilde{P})$ should be a rational multiple of $L^*(H^{n-1}(X), 0)$. For example, Deninger [19, Page 273] and Bornhorn [4, Theorem 2.2] have shown that for every $k \in \mathbb{Z} \setminus \{-4, 0, 4\}$ these hypotheses are satisfied by the Laurent polynomial appearing in (4.11), which was originally discovered by Boyd [8, Equation (1-32)].

Successively exact polynomials

What happens if these hypotheses are not satisfied? In some cases, one can still get relations between Mahler measures and special values of L-functions. For example, suppose that γ_P is not closed, but that the hypothesis $\gamma_P \subseteq V_P^{\text{reg}}(\mathbb{C})$ still holds true. Suppose furthermore that η_n is *exact* on V_P^{reg}. In this case, we can combine (4.15) with Stokes's theorem to see that

$$m(P) - m(\tilde{P}) = \frac{(-1)^n}{(2\pi i)^{n-1}}\int_{\gamma_P}\eta_n = \frac{(-1)^n}{(2\pi i)^{n-1}}\int_{\partial\gamma_P}\omega,$$

where ω is any primitive of the restriction of η_n to V_P^{reg}. This allows us to relate the Mahler measure of P to an integral of a differential form having smaller degree, which in turn can again be seen as a regulator integral, thanks to an insight of Maillot [10, Section 8], which was later developed in depth by Lalín [42].

More precisely, one can observe that

$$\partial\gamma_P \subseteq V_P(\mathbb{C}) \cap \mathbb{T}^n \subseteq W_P(\mathbb{C}),$$

where $W_P := V_P \cap V_{P^*}$ is the intersection between the hypersurface defined by P and by its reciprocal P^*. Symmetrizing ω with respect to the map $\mathbb{G}_m^n \to \mathbb{G}_m^n$ given by inversion of coordinates we can assume that the restriction of ω to W_P^{reg} is *closed*. Now, suppose that:

- the boundary of Deninger's cycle γ_P defined in (4.13) does not meet any singular point of W_P, *i.e.* $\partial\gamma_P \subseteq W_P^{\text{reg}}(\mathbb{C})$;

- there exists a smooth and projective variety Y and a motivic cohomology class $\beta \in H^{n-1}_{\mathcal{M},\mathbb{Z}}(Y;\mathbb{Q}(n))$ such that $W_P^{\mathrm{reg}} \subseteq Y$ and $\mathrm{reg}_\infty(\beta|_{W_P^{\mathrm{reg}}}) = [\omega] \in H^{n-1}_{\mathcal{D}}(W_P^{\mathrm{reg}};\mathbb{R}(n)) \cong H^{n-2}_{\mathrm{dR}}(W_P^{\mathrm{reg}};\mathbb{R}(n-1))$.

Then, we see as before that $m(P) - m(\tilde{P})$ appears as one of the entries of a matrix which represents the Beilinson regulator integral $\mathrm{reg}_\infty \colon H^{n-1}_{\mathcal{M},\mathbb{Z}}(Y;\mathbb{Q}(n)) \to H^{n-1}_{\mathcal{D}}(Y_\mathbb{R};\mathbb{R}(n))$. Therefore, if $\dim_\mathbb{R}(H^{n-1}_{\mathcal{D}}(Y_\mathbb{R};\mathbb{R}(n))) = 1$, then Beilinson's conjecture implies that $m(P) - m(\tilde{P})$ should be a rational multiple of the special value $L^*(H^{n-2}(Y),-1)$, which corresponds to the special value $L(H^{n-2}(Y),n)$ under the conjectural functional equation. When $n = 2$, Guilloux and Marché [30] have called this type of polynomials *exact*, in view of the exactness of the differential form $\eta_2 = \log|x_1| d\arg(x_2) - \log|x_2| d\arg(x_1)$. We believe that it is reasonable to keep this terminology in any number of variables.

On the other hand, it may happen that γ_P is closed, but that $\gamma_P \not\subseteq V_P^{\mathrm{reg}}(\mathbb{C})$. In this case, one can use (4.15) in order to write

$$m(P) - m(\tilde{P}) = \frac{(-1)^n}{(2\pi i)^{n-1}} \int_{\tilde{\gamma}_P} \tilde{\eta}_n,$$

where $\tilde{\eta}_n$ is the pull-back of η_n to a desingularization $\tilde{V}_P$ of V_P, and $\tilde{\gamma}_P$ is the strict transform of γ_P to $\tilde{V}_P$. Note that the cycle $\tilde{\gamma}_P$ is *not* closed, in general. However, since we are assuming that γ_P is closed, the boundary of $\tilde{\gamma}_P$ is going to be contained in the exceptional divisor $Z \subseteq \tilde{V}_P$. Furthermore, if we find that:

- $\tilde{\eta}_n$ is exact on $\tilde{V}_P$;
- the boundary of the strict transform $\tilde{\gamma}_P$ does not meet any singular point of Z, *i.e.* $\partial\tilde{\gamma}_P \subseteq Z^{\mathrm{reg}}(\mathbb{C})$,

we are able to write $m(P) - m(\tilde{P})$ as a regulator integral over a smooth compactification of Z^{reg}.

The situations described in the two previous paragraphs can in general be combined. Suppose for example that γ_P is not closed, and that P is exact. Then we can write $m(P) - m(\tilde{P})$ as an integral of a differential form ω, defined as a primitive of the restriction of η_n to V_P^{reg}, over the *closed* topological chain $\delta_P := \partial\gamma_P \subseteq W_P(\mathbb{C})$. If this topological chain touches some of the singular points of W_P, we can then write $m(P) - m(\tilde{P})$ as an integral of a new differential form $\tilde{\omega}$ over a new topological chain $\tilde{\delta}_P$ inside a desingularization $\tilde{W}_P$. Since the new chain $\tilde{\delta}_P$ is not necessarily closed, it may happen that $\tilde{\omega}$ is exact, and in this case we may be able to write $m(P) - m(\tilde{P})$ as a regulator integral over $\partial\tilde{\delta}_P$, which is contained in the exceptional locus $Z \subseteq \tilde{W}_P$. It may happen once again that the closed topological chain $\partial\tilde{\delta}_P$ touches the singular points of Z, which would allow us to consider a new desingularization $\tilde{Z}$ and iterate the process. There is a more systematic way of doing this, provided by the following result.

Theorem 4.16 (Brunault & P.). *For every polynomial $P \in \mathbb{C}[x_1,\dots,x_n] \setminus \{0\}$, there exists a singular homology class $\beta_P \in H_{n-1}(V_{PP^*}(\mathbb{C});\mathbb{Z})$ such that*

$$m(P) - m(\tilde{P}) = \frac{(-1)^n}{(2\pi i)^{n-1}} \langle \mathrm{reg}_\infty(\{x_1,\dots,x_n\}|_{V_{PP^*}}), \beta_P \rangle_{V_{PP^*}},$$

where $\langle \cdot,\cdot \rangle_{V_{PP^*}}$ is the period pairing on V_{PP^*}.

In other words, the previous result shows how one can write $m(P) - m(\tilde{P})$ as a regulator integral over the variety $V_{PP^*} = V_P \cup V_{P^*}$. Note that, if $n \geq 2$, this variety is never projective, and is never smooth unless $V_P = V_{P^*}$, which would mean that P is essentially reciprocal. Therefore, to analyze the relation between regulator integrals on V_{PP^*} and special values of L-functions, one has to relate V_{PP^*} to *several* smooth and projective varieties. To do so, one can observe that, thanks to Hironaka's theorem on the existence of resolutions of singularities in characteristic zero [33], there exists a smooth variety U, endowed with a divisor $D \subsetneq U$ having simple normal crossings, and with a map $U \to V_{PP^*}$ that resolves the singularities of V_{PP^*} In particular, this map induces an isomorphism between $U \setminus D$ and $V_{PP^*}^{\text{reg}}$. Moreover, thanks again to the theory of resolutions of singularities, we know that there exists a smooth and projective variety X, endowed with two divisors $A, B \subsetneq X$ such that $A \cup B$ has simple normal crossings, and $X \setminus A \cong U$, while we have that $B \setminus (A \cap B) \cong D$. Thanks to several good properties satisfied by singular homology and motivic cohomology, one may show that the motivic cohomology class $\alpha_P := \{x_1, \dots, x_n\}|_{V_{PP^*}} \in H^n_{\mathcal{M}}(V_{PP^*}; \mathbb{Q}(n))$ induces a class $\tilde{\alpha}_P \in H^n_{\mathcal{M}}(X \setminus A, B \setminus (A \cap B); \mathbb{Q}(n))$, while the singular homology class $\beta_P \in H_{n-1}(V_{PP^*}(\mathbb{C}); \mathbb{Z})$ induces another class $\tilde{\beta}_P \in H_{n-1}(X(\mathbb{C}) \setminus A(\mathbb{C}), B(\mathbb{C}) \setminus (A(\mathbb{C}) \cap B(\mathbb{C})); \mathbb{Z})$, in such a way that

$$m(P) - m(\tilde{P}) = \frac{(-1)^n}{(2\pi i)^{n-1}} \langle \tilde{\alpha}_P, \tilde{\beta}_P \rangle_{(X \setminus A, B \setminus (A \cap B))},$$

as implied by Theorem 4.16. Therefore, we see that $m(P) - m(\tilde{P})$ appears as a Deligne period of the relative motivic cohomology group $H^n_{\mathcal{M}}(X \setminus A, B \setminus (A \cap B); \mathbb{Q}(n))$, which can be computed by the means of two *spectral sequences*. More precisely, write $\{A_1, \dots, A_r\}$ for the components of A, and $\{B_1, \dots, B_s\}$ for the components of B. Then, for every pair of sets $I \subseteq \{1, \dots, r\}$ and $J \subseteq \{1, \dots, s\}$ we can define the varieties $A_I := \bigcap_{i \in I} A_i$ and $B_J := \bigcap_{j \in J} B_j$, with $A_\emptyset := X$ and $B_\emptyset := X$. It turns out that each variety $A_I \cap B_J$ is smooth and projective, because $A \cup B$ is supposed to be a divisor with simple normal crossings inside the smooth and projective variety X. Moreover, we have the weight and relative cohomology spectral sequences:

$$\bigoplus_{|J|=p} H^q_{\mathcal{M}}(B_J; \mathbb{Q}(n)) \Rightarrow H^{p+q}_{\mathcal{M}}(X, B; \mathbb{Q}(n))$$
$$\bigoplus_{|I|=-p} H^{q+2p}_{\mathcal{M}}(A_I; \mathbb{Q}(n+p)) \Rightarrow H^{p+q}_{\mathcal{M}}(X \setminus A; \mathbb{Q}(n)),$$

which can be used under the additional constraint $p + q = n$ to compute the motivic cohomology group $H^n_{\mathcal{M}}(X \setminus A, B \setminus (A \cap B); \mathbb{Q}(n))$. In particular, one can construct a *bi-filtration* on this motivic cohomology group, whose associated graded quotients are all the motivic cohomology groups

$$\{H^{n-|I|-|J|}_{\mathcal{M}}(A_I \cap B_J; \mathbb{Q}(n - |I|)) \colon I \subseteq \{1, \dots, r\},\ J \subseteq \{1, \dots, s\}\}, \tag{4.17}$$

which implies that the special values of L-functions associated to smooth and projective varieties which could possibly be related to $m(P) - m(\tilde{P})$ should lie in the set

$$\{L^*(H^{n-1-|I|-|J|}(A_I \cap B_J), -|J|) \colon I \subseteq \{1, \dots, r\},\ J \subseteq \{1, \dots, s\}\},$$

at least according to Beilinson's conjecture.

Using this framework, we say that a polynomial $P \in \mathbb{Q}[x_1, \dots, x_n]$ is *k-exact* if the pure components $\alpha_{P,(I,J)} \in H^{n-|I|-|J|}_{\mathcal{M}}(A_I \cap B_J; \mathbb{Q}(n - |I|))$ of the motivic cohomology class $\tilde{\alpha}_P \in H^n_{\mathcal{M}}(X \setminus A, B \setminus (A \cap B); \mathbb{Q}(n))$ mentioned above vanish whenever $|I| + |J| < k$. In other words, this means that the only non-trivial pure components of $\tilde{\alpha}_P$ live on smooth and projective varieties whose dimension differs from the expected dimension, which is $n - 1$, by at least k units. Morally speaking, when a polynomial is k-exact one has to take primitives and residues of the class $\tilde{\alpha}_P$ at least k-times, in order to get to the desired smooth and projective varieties $A_I \cap B_J$ on which the components $\alpha_{P,(I,J)}$ are non-trivial.

A gallery of examples

This geometric method is sufficient to explain many, if not essentially all of the existing identities between Mahler measures and special values of L-functions. For example, if one looks at the polynomials

$$P_k := x_1 + x_1^{-1} + x_2 + x_2^{-1} + k,$$

featured in Boyd's conjectural relation (4.11), then for every integer $k \in \mathbb{Z} \setminus \{-4, 0, 4\}$ the motivic cohomology class $\alpha_{P_k} \in H^2_{\mathcal{M}}(X_k \setminus A_k, B_k \setminus (A_k \cap B_k); \mathbb{Q}(2))$ lies in fact in the group $H^2_{\mathcal{M},\mathbb{Z}}(X_k; \mathbb{Q}(2))$, and we have that $X_k \cong E_k \colon y^2 + kxy = x^3 - 2x^2 + x$. However, when $k = \pm 4$ then X_k is a nodal cubic, and the class $\alpha_{P_{\pm 4}}$ comes from $H^1_{\mathcal{M},\mathbb{Z}}(B_1; \mathbb{Q}(2))$, where $B = B_1$ is the exceptional divisor, which consists of two points defined over $\mathbb{Q}(i)$.

On the other hand, if we look at the polynomials

$$P_k = x_1 + x_2 + k,$$

we see that when $k \geq 3$ the class α_{P_k} comes from $H^1_{\mathcal{M}}(A_1; \mathbb{Q}(1))$, which corresponds to the easily proven identity $m(P_k) = \log|k|$. However, when $k = 1$ the polynomial $P_1 := x_1 + x_2 + 1$, featured in Smyth's Proposition 4.9, is actually exact, and the class α_{P_1} comes from $H^1_{\mathcal{M},\mathbb{Z}}(B_1; \mathbb{Q}(2))$, where $B = B_1$ is a union of two points defined over $\mathbb{Q}(\sqrt{-3})$.

We can moreover have identities of mixed type. For example, if we consider the polynomials

$$P_k := x_2^2 + k(x_1 + 1)x_2 + x_1^3,$$

that were already featured in Boyd's conjectural identity (4.12), we see that the class α_{P_k} splits into two components, one belonging to $H^2_{\mathcal{M},\mathbb{Z}}(X_k; \mathbb{Q}(2))$, where X_k is an elliptic curve, and the other belonging to $H^1_{\mathcal{M}}(A_{1,k}; \mathbb{Q}(1))$, which yields the logarithmic factor appearing in (4.12). On the other hand, if we look at the polynomial

$$P = (x_1^2 + 1)^2 x_2^2 + 2x_1 x_2 + 1,$$

considered by Bornhorn [4, Section 4], we see that the class α_P splits again as an element of the group $H^2_{\mathcal{M}}(X; \mathbb{Q}(2))$, where $X \colon y^2 = x^3 - x^2 - 4x + 4$ is an elliptic curve, and an element of $H^1_{\mathcal{M}}(B_1; \mathbb{Q}(2))$, where $B = B_1$ is the union of two points defined over $\mathbb{Q}(\sqrt{-3})$. This corresponds to the conjectural identity

$$m((x_1^2 + 1)^2 x_2^2 + 2x_1 x_2 + 1) \stackrel{?}{=} L'(E, 0) + L'(\chi_{-3}, -1),$$

which was again found by Boyd [8, Page 78].

In three variables, the situation becomes even more varied. For example, if one considers the family of polynomials

$$P_k := x_1 + x_1^{-1} + x_2 + x_2^{-1} + x_3 + x_3^{-1} + k,$$

one has again that the class α_k associated to P_k lies in the top cohomology $H^3_{\mathcal{M},\mathbb{Z}}(X_k; \mathbb{Q}(3))$, giving a relation between $m(P_k)$ and the L-functions associated to the K3-surfaces X_k which are birational to V_{P_k}. On the other hand, if one considers the polynomial

$$P = x_1 + x_2 + x_3 + 1,$$

whose Mahler measure, featured in (4.10), was computed by Smyth, the corresponding class α_P decomposes in the motivic cohomology groups appearing in (4.17) that correspond to the pairs of sets given by

$(I,J) \in \{(\emptyset,\{1,2\}),(\emptyset,\{2,3\}),(\emptyset,\{1,3\})\}$. In other words, in this case α_P is concentrated on the two-fold intersections of the three components of the exceptional divisor $B = B_1 \cup B_2 \cup B_3$, which are points defined over $\mathbb{Q}$, corresponding to the identity (4.10) proven by Smyth. Therefore, this polynomial fits into the group of 2-*exact polynomials*. On the other hand, there are some very interesting examples of 1-exact polynomials in three variables, whose Mahler measures turn out to be related to elliptic curves. For example, the polynomial

$$P := (x_1+1)(x_2+1)+x_3,$$

which was found by Boyd and Rodriguez-Villegas [10, Section 8] and studied by Lalín [43], has a corresponding motivic cohomology class α_P which lies entirely in $H^2_{\mathcal{M}}(B_1;\mathbb{Q}(3))$, where $B = B_1$ is isomorphic to $E\colon y^2+xy+y = x^3+x^2$. This corresponds to the relation

$$m((x_1+1)(x_2+1)+x_3) = -2\cdot L'(E,-1), \tag{4.18}$$

which was found by Boyd and Rodriguez-Villegas [10, Section 8] and was recently proven by Brunault [11], following previous work of Lalín [43], who proved that Beilinson's conjecture implies that (4.18) holds up to a rational number. Finally, we can find "spurious" identities also for Mahler measures in three variables. For example, if

$$P = (x_1-1)^3+(x_1+1)^3(x_2+x_3),$$

then α_P splits in those motivic cohomology groups appearing in (4.17) associated to the pairs of sets given by $(I,J) \in \{(\emptyset,\emptyset),(\emptyset,\{1,2\})\}$, which corresponds to the identity

$$m((x_1-1)^3+(x_1+1)^3(x_2+x_3)) = -2\cdot L'(E,-1)+112\cdot\zeta'(-2),$$

recently shown by Trieu [72].

Finally, this type of interesting relations between Mahler measures and special values of L-functions continue to appear, in even more complicated fashions, when one considers polynomials in four or more variables. For example, one can consider again the reciprocal family

$$P_k := x_1+x_1^{-1}+x_2+x_2^{-1}+x_3+x_3^{-1}+x_4+x_4^{-1}+k,$$

whose corresponding class α_{P_k} is genreically concentrated in the biggest cohomology group $H^4_{\mathcal{M}}(X_k;\mathbb{Q}(4))$, where X_k is the Calabi-Yau threefold birational to V_{P_k}. On the other hand, if

$$P = (x_1-1)(x_2-1)-(x_3-1)(x_4-1),$$

then the class α_P is concentrated in the group corresponding to $(I,J) = (\{1\},\{1,2\})$, which matches with the identity

$$m((x_1-1)(x_2-1)-(x_3-1)(x_4-1)) = -18\cdot\zeta'(-2),$$

proven by D'Andrea and Lalín [17]. Therefore, the class corresponding to this polynomial is completely supported "at infinity", and within this group the resulting class is 2-exact. On the other hand, if one considers the polynomial

$$P = 1+x_1+x_2+x_3+x_4-x_1x_2,$$

the corresponding class α_P turns out to be supported on $(I,J) = (\emptyset,\{1\})$, where $B = B_1$ is a K3 surface. This corresponds to the numerical identity

$$m(1+x_1+x_2+x_3+x_4-x_1x_2) \stackrel{?}{=} -7\cdot L'(f,-1),$$

where $f = q\prod_{n=1}^{+\infty}(1-q^n)^2(1-q^{2n})(1-q^{4n})(1-q^{8n})^2$ is a cusp form of weight three and level $\Gamma(8)$. Finally, if we look at the polynomial

$$P := x_1x_2x_4 + x_1x_3x_4 + x_2x_3x_4 + x_1x_2 + x_1x_3 - x_2x_3 - x_2x_4 + x_3x_4 - x_2 + x_3 - x_4 + 1$$

recently found by Brunault and the author of the present notes, we see that α_P is concentrated on the group corresponding to $(I, J) = (\emptyset, \{1, 2\})$, where $B_1 \cap B_2$ is birational to the elliptic curve $E\colon y^2 = x^3 - x$. This corresponds to the conjectural identity

$$m(x_1x_2x_4 + x_1x_3x_4 + x_2x_3x_4 + x_1x_2 + x_1x_3 - x_2x_3 - x_2x_4 + x_3x_4 - x_2 + x_3 - x_4 + 1) \stackrel{?}{=} \frac{1}{6} \cdot L'(E, -4) \quad (4.19)$$

which was found by computing numerically, to very high accuracy, the Mahler measure appearing in (4.19). To do so, Ringeling and the author of the present notes have used the following general identity

$$m(P) = \frac{1}{2}\left(\log(k) - \int_0^{1/k}\left(\frac{1}{(2\pi i)^n}\int_{\mathbb{T}^n}\frac{Q(z_1,\dots,z_n)}{1 - t\cdot Q(z_1,\dots,z_n)}\frac{dz_1}{z_1}\cdots\frac{dz_n}{z_n}\right)dt\right),$$

where k denotes the constant coefficient of $P \cdot P^*$, while $Q := P \cdot P^* - k$. This identity presents $m(P)$ as a *Kontsevich-Zagier period*, and allows one to compute $m(P)$ as the value at $T = 1/k$ of the *period function*

$$T \mapsto \frac{1}{2}\left(\log(k) - \int_0^{T}\left(\frac{1}{(2\pi i)^n}\int_{\mathbb{T}^n}\frac{Q(z_1,\dots,z_n)}{1 - t\cdot Q(z_1,\dots,z_n)}\frac{dz_1}{z_1}\cdots\frac{dz_n}{z_n}\right)dt\right)$$

which satisfies a linear differential equation with polynomial coefficients. Such a differential equation can be computed using either creative telescoping [15] or an algorithm due to Lairez [40], which finally allows to check identities such as (4.19) with a very small error.

What about five or more variables? The Mahler measure of the following family of polynomials

$$P_n := x_1(x_2 - 1)\cdots(x_n - 1) + (x_2 + 1)\cdots(x_n + 1),$$

has been computed by Lalín [41] to be a rational linear combination of the zeta values $\{\zeta'(-2), \dots, \zeta'(-2m)\}$ if $n = 2m + 1$, and of the Dirichlet L-values $\{L'(\chi_{-4}, -1), \dots, L'(\chi_{-4}, 1 - 2m)\}$ if $n = 2m$. Geometrically, one can show that the pure components $\alpha_{P_n,(I,J)}$ vanish unless $|I| + |J| = n - 1$. Moreover, the components of the divisors A_{P_n} are given by the linear subspaces $\{x_1 = 0, x_j = -1\}$ for $j \in \{2, \dots, n\}$, and by the varieties

$$\left\{x_k = 0, \prod_{\substack{h=2,\dots,n\\ h\neq k}}(x_h + 1) + x_1\prod_{\substack{h=2,\dots,n\\ h\neq k}}(x_h - 1) = 0\right\},$$

for $k \in \{2, \dots, n\}$, as well as by the components at infinity $\{x_0 = 0, x_i = 0\}$, for $i \in \{1, \dots, n\}$, where x_0 denotes the homogenizing variable. Finally, the components of B_n correspond to the linear subspaces $\{x_i = 1, x_j = -1\}$, for $i, j \in \{2, \dots, n\}$, and by the variety $\{P_n = 0, x_1^2 = (-1)^{n-1}\}$. We see immediately that if n is odd then the quadric $x_1^2 = (-1)^{n-1}$ splits into two rational hyperplanes, yielding to the L-values $\{\zeta'(-2), \dots, \zeta'(-2m)\}$, while if n is even then this quadric gives rise to some points defined over $\mathbb{Q}(\sqrt{-1})$, and therefore to the L-values $\{L'(\chi_{-4}, -1), \dots, L'(\chi_{-4}, 1 - 2m)\}$.

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